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THE UNIFORMIZATION THEOREM FOR COMPACT KÄHLER MANIFOLDS OF NONNEGATIVE HOLOMORPHIC BISECTIONAL CURVATURE

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Regarding the uniformization problem on compact Kähler manifolds of arbitrary dimensions, a very fundamental contribution was made by Mori [20] and Siu-Yau [24] when they affirmed the Frankel conjecture. They proved

Theorem (*special case of Mori* [20] *and Siu-Yau* [24]). *Let X be a compact Kähler manifold of positive holomorphic bisectional curvature. Then X is biholomorphic to the complex projective space $\mathbb{P}^n$.*

The theorem of Mori is stronger since he only assumed, in case the ground field is $\mathbb{C}$, that X is a compact complex manifold with ample tangent bundle and since his theorem applies equally to any algebraically closed field of arbitrary characteristic. Returning to the situation of compact Kähler manifold, it was natural to conjecture

(W_n) *Weak form of the uniformization conjecture for manifolds of nonnegative curvature*. Let X be an n-dimensional compact Kähler manifold of nonnegative holomorphic bisectional curvature and let $\tilde{X}$ be its universal covering space. Then $\tilde{X}$ is biholomorphic to $\mathbb{C}^k \times M$ for some nonnegative integer $k \leqslant n$ and some compact Hermitian symmetric manifold M.

In fact, (W_n) was in essence conjectured by Siu-Yau [24] prior to the solutions of the Frankel conjecture as part of a program to study the uniformization problem in higher dimensions. The purpose of the present article is to give a proof of the following stronger form of the uniformization conjecture for manifolds of nonnegative curvature, to be denoted by (S_n).

Main Theorem. *Let (X, h) be an n-dimensional compact Kähler manifold of nonnegative holomorphic bisectional curvature and let $(\tilde{X}, \tilde{h})$ be its universal covering space. Then there exist nonnegative integers k, $N_1, \cdots, N_l$ and irreducible compact Hermitian symmetric spaces $M_1, \cdots, M_k$ of rank $\geqslant 2$, such that*

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$(\tilde{X}, \tilde{h})$ is isometrically biholomorphic to

$$(\mathbb{C}^k, g_0) \times (\mathbb{P}^{N_1}, \theta_1) \times \cdots \times (\mathbb{P}^{N_l}, \theta_l) \times (M_1, g_1) \times \cdots \times (M_p, g_p),$$

where g_0 denotes the Euclidean metric on $\mathbb{C}^k$, $g_1, \cdots, g_p$ are canonical metrics on $M_1, \cdots, M_p$, and θ_i, $1 \leqslant i \leqslant l$, is a Kähler metric on $\mathbb{P}^{N_i}$ carrying nonnegative holomorphic bisectional curvature.

In the years 1980–1985 there has been a lot of work related to (W_n) and (S_n). In 1980, Siu [22] obtained a curvature characterization for hyperquadrics. Using Siu's result and the evolution equation of Hamilton [8], Bando [1] solved (W_3) in 1983, the case of (W_2) being known to Howard-Smyth [11] much earlier. On the other hand, Mok-Zhong [18], [19] studied (S_n) with the additional assumption that (X, h) is Einstein and showed in 1984 that (X, h) is isometrically a Hermitian locally symmetric manifold. Furthermore, Mok [17] showed very recently that (W_n) implies (S_n), while Cao-Chow [5] showed that (S_n) holds under the stronger assumption that the curvature operator is semipositive in the dual Nakano sense.

It is well known, using the splitting theorem of Howard-Smyth-Wu [12], that one can reduce (S_n) to the proof of the following special case.

Theorem 1. *Let (X, h) be a compact Kähler manifold of nonnegative holomorphic bisectional curvature such that the Ricci curvature is positive at one point. Suppose the second Betti number of X is equal to one. Then either X is biholomorphic to the complex projective space or (X, h) is isometrically biholomorphic to an irreducible compact Hermitian symmetric manifold of rank $\geqslant 2$.*

Our proof of Theorem 1 depends in essential ways on Mori's solution to the Frankel conjecture [20], the evolution equation of Hamilton [8], the methods of Mok-Zhong [19] for the Einstein case, and the characterization of locally symmetric spaces by their holonomy groups of Berger [4]. Because we will need Mori's theory of rational curves, our proof is not completely transcendental in nature. If fact, it was essential in Mori [20] to use algebraic geometry over fields of characteristic > 0. At present, completely within the realm of transcendental methods, one can adapt the methods of Siu-Yau [24] and Siu [22] using stable harmonic maps to give a proof of Theorem 1 only with an additional nondegeneracy condition on the curvature tensor, namely, condition (C) of Siu [23], a condition satisfied by all irreducible compact Hermitian symmetric manifolds. Eventually, it would be desirable to remove this condition to give a complete proof of Theorem 1 using transcendental methods.

In order to explain our approach, it is helpful to discuss the geometry of compact Hermitian symmetric manifolds and particularly the role played by certain special rational curves (cf. Mok [16]). Let (M, g) be an irreducible compact Hermitian symmetric manifold and let σ: $(M, g) \hookrightarrow$ ($\mathbb{P}^N$, Fubini-Study) be the first canonical (isometric) projective embedding of M. Let Σ be the collection of projective lines in $\mathbb{P}^N$ that are already contained in M (which we identify with $\sigma(M)$); they are totally geodesic in (M, g). Moreover, a unit vector $\alpha \in T_x(M)$ realizes the global maximum of holomorphic sectional curvature if and only if α is tangent to some projective line passing through x. Let $\mathscr{S}_M \subset \mathbb{P}T_M$ be given by the set of all possible tangent directions to such S. Then $\mathscr{S}_M$ is a complex analytic submanifold of $\mathbb{P}T_M$ fibered over M. It is invariant under parallel transport on (M, g) because holomorphic sectional curvatures are invariant under parallel transport on a symmetric space. Let $S \in \Sigma$ and $T_x(S) = \mathbb{C}\alpha$; then, according to Mok-Zhong [19] there is an orthogonal decomposition $T_x(M) = \mathbb{C}\alpha + \mathscr{H}_\alpha + \mathscr{N}_\alpha$ into eigenspaces of the Hermitian form $\mathscr{H}_\alpha(\chi, \eta) = R_{\alpha\bar{\alpha}\chi\bar{\eta}}$ such that $\mathscr{H}_\alpha$ corresponds to the eigenvalue $\frac{1}{2}R_{\alpha\bar{\alpha}\alpha\bar{\alpha}}$ while $\mathscr{N}_\alpha$ corresponds to the eigenvalue zero. On S we have the decomposition $T_{M|S} \cong \mathscr{O}(2) \oplus \mathscr{O}(1)^{\dim \mathscr{H}_\alpha} \oplus \mathscr{O}^{\dim \mathscr{N}_\alpha}$. The dimensions of $\mathscr{H}_\alpha$ and $\mathscr{N}_\alpha$ are independent of the choice of $S \in \Sigma$ and $x \in M$. Moreover, $1 + \dim \mathscr{N}_\alpha$ is equal to the degree of (strong) nondegeneracy of bisectional curvatures of (M, g) as defined in Siu [23]. We note that $\mathscr{S}_M$ is independent of the choice of the special canonical metric g (for any $\Phi \in \mathrm{Aut}(M)$, Φ^*g is a canonical metric on M) since biholomorphic transformations of M are restrictions of projective linear transformations of $\mathbb{P}^N$, which preserve the space Σ of projective lines.

Our idea is to recapture the variety $\mathscr{S}_M \subset \mathbb{P}T_M$. $\mathscr{S}$ is on the one hand an analytic object which can be constructed by deforming a rational curve in M of minimal degree; on the other hand a differential-geometric object invariant under parallel transport. Now, given a compact Kähler manifold (X, h) of nonnegative bisectional curvature and positive Ricci curvature, Mori's theory of rational curves [20] will allow us to construct an object $\mathscr{S} \subset \mathbb{P}T_X$ similar to the $\mathscr{S}_M$ described above. However, in order to prove Theorem 1 we have to consider (X, h) of nonnegative bisectional curvature everywhere and positive Ricci curvature at one point. To start with we are going to use the evolution equation of Hamilton [8] to deform h.

In 1982, Hamilton [8] studied compact three-manifolds with positive Ricci curvature by using the equation $\frac{d}{dt}g_{ij} = -R_{ij}$ (respectively the metric and Ricci tensors) in a suitably normalized form and showed that such metrics can be deformed to one of constant sectional curvature. In 1984 Bando [1] applied the strong maximum principle of Hamilton [8] to the situation of (complex) three-dimensional compact Kähler manifolds X of nonnegative bisectional curvature. He proved that the nonnegativity is preserved in the evolution. In higher dimensions it has been an open problem if the nonnegativity of curvatures (in some sense) is preserved by the evolution equation of Hamilton [8]. Very recently, Hamilton [8] affirmed this in the case of compact Riemannian manifolds with nonnegative curvature operator and used it to classify such manifolds in case of dimension 4. His method was used by Cao-Chow [5] to affirm (S_n) under the stronger assumption of nonnegativity of the curvature operator. In this article, we show somewhat surprisingly that the class of Kähler metrics of nonnegative bisectional curvature on compact manifolds is preserved under the evolution equation. Furthermore, if the initial metric is of positive Ricci curvature at one point, then the evolved metric is of positive Ricci curvature and positive holomorphic sectional curvature. Essentially, by Hamilton [8] this reduces to proving that some tensor $F_{\alpha\bar{\alpha}\beta\bar{\beta}}$ quadratic in the curvature R verifies $F_{\alpha\bar{\alpha}\beta\bar{\beta}} \geqslant 0$ whenever $R_{\alpha\bar{\alpha}\beta\bar{\beta}} = 0$ for an evolved metric. Such an inequality in the Einstein case was obtained by Mok-Zhong [19] for certain zeros of bisectional curvature. The theoretical relationship between algebraic problems in the Einstein case, as tackled by Mok-Zhong [19] and similar algebraic problems in the general case will be explained in detail in §1.

From now on we consider exclusively evolved metrics h. Since (X, h) carries positive Ricci curvature, by the results of Mori [20] we obtain a rational curve $f\colon \mathbb{P}^1 \to X$ such that $f^*(K_X^{-1}) \cong \mathcal{O}(q)$ with $0 < q \leqslant n + 1$. We consider the minimal such integer q. Let $\mathscr{C}_0$ be the Chow variety of all cycles C such that $K_X^{-1} \cdot C = q$ and let $\mathscr{C}$ be the irreducible component of $\mathscr{C}_0$ containing $[f(\mathbb{P}^1)]$. Suppose $q < n + 1$. Then, generically $f^*(T_X) \cong \mathcal{O}(a_1) \oplus \cdots \oplus \mathcal{O}(a_k) \oplus \mathcal{O}^l$ with $a_1 \geqslant \cdots \geqslant a_k > 0$ and $l > 0$. This implies that, for a smooth point P of $f(\mathbb{P}_1)$ with $T_p(f(\mathbb{P}^1)) = \mathbb{C}\alpha$, $R_{\alpha\bar{\alpha}\beta\bar{\beta}} = 0$ for β in a $\mathbb{C}$-linear subspace of $T_p(X)$ of at least l dimensions. We define $\mathscr{S} \subset \mathbb{P}T_X$ to be the closure of the set of all tangent directions at smooth points to cycles $C \in \mathscr{C}$. In case $q = n + 1$, by the criterion of Mori either X is biholomorphic to $\mathbb{P}^n$ or there exists for every $P \in X$ some rational curve $f\colon \mathbb{P}^1 \to X$ such that $f^*(T_X)$ has some trivial components. Let $\mathscr{C}' \subset \mathscr{C}$ be the subspace consisting of such rational curves and consider $\mathscr{S}$ obtained from $\mathscr{C}'$ in a similar way. (Every cycle $C \in \mathscr{C}$ is an irreducible rational cycle by the minimality of q.) By studying the evolution equation for bisectional curvatures we showed that $\mathscr{S}$ is invariant under parallel transport. Essentially this is obtained by studying the two decompositions of $T_x(X)$ coming from (i) the splitting of $f^*(T_X)$ into line bundles, (ii) the splitting of $T_x(X)$ into eigenspaces of the Hermitian form $\mathscr{H}_\alpha(\chi, h) = R_{\alpha\bar{\alpha}\chi\bar{\eta}}(h)$. After proving the invariance of $\mathscr{S}$ under parallel transport it remains to apply Berger's characterization of locally symmetric spaces by the holonomy group [4] to conclude the proof of Theorem 1 and hence our uniformization theorem for compact Kähler manifolds of nonnegative bisectional curvature.

A preliminary version of the present article with an outline of proof has appeared in Mok [17].

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CONTENTS

1. The evolution equation of Hamilton—preservation of the nonnegativity of bisectional curvature

(1.1) Background and statement of results. In [8] Hamilton studied the evolution equation $\frac{\partial}{\partial t} g_{ij} = -R_{ij}$ (respectively the metric and the Ricci tensors) on compact three-manifolds of positive Ricci curvature and showed that such metrics can be deformed to one of constant sectional curvature after suitably rescaling the evolved metrics. The starting point of his analysis was his strong maximum principle on tensors. Thus, he proved short time solvability of the equation in arbitrary dimensions and showed that in three real dimensions the evolution of metrics preserves nonnegativity of Ricci curvature. In the Kähler case, the evolved metrics $g_{i\bar{j}}(t)$ remain Kähler, as remarked in Bando [1]. Using the strong maximum principle of Hamilton [18], we prove

Proposition (1.1). *Let $(X, g_{i\bar{j}})$ be a compact Kähler manifold of nonnegative holomorphic bisectional curvature. Then, the evolved metrics $g_{i\bar{j}}(t)$, $t > 0$, also carry nonnegative holomorphic bisectional curvature. Moreover, if $(x, g_{i\bar{j}})$ is of positive Ricci curvature at one point, then the evolved metrics are of positive holomorphic sectional curvature and positive Ricci curvature everywhere.*

The special case of Proposition (1.1) for three-dimensional compact Kähler manifolds was proved by Bando [1]. The positivity of holomorphic sectional curvatures was not stated there but follows easily from the proof. To explain our method we recall first of all the strong maximum principle of Hamilton [8] for tensors, formulated here for Kähler manifolds as was done in Bando [1].

Proposition (*Bando* [1, §4, *Proposition* 1]). *Let* $(X, g_{i\bar{j}})$ *be an n-dimensional compact Kähler manifold. Let u denote a tensor field which has the same type and symmetric properties as the curvature tensor. Let* $(X, g_{i\bar{j}}(t))$, $0 < t < \varepsilon$, *denote the evolved metric given by Hamilton's equation* $\frac{\partial}{\partial t} g_{i\bar{j}} = -R_{i\bar{j}}$. *Denote by* Δ *the tensor Laplacian (depending on the Kähler metric), and suppose the tensor u evolves according to the equation*

$$\frac{\partial u}{\partial t} = \Delta u + F(u),$$

where F satisfies the null-vector condition

(#) *If* $u \geqslant 0$ *and there exist two nonzero vectors* $\alpha, \zeta \in T_x X$ *such that* $u(\alpha, \bar{\alpha}; \zeta, \bar{\zeta})(x) = 0$, *then* $F(u)(\alpha, \bar{\alpha}, \zeta, \bar{\zeta})(x) \geqslant 0$.

If the initial u is nonnegative, then it remains so. Moreover, if the initial u is positive somewhere, then the solution u is positive everywhere for $t > 0$.

Remark. Here we write $u \geqslant 0$ to mean $u(\chi, \bar{\chi}; \eta, \bar{\eta}) \geqslant 0$ for all χ, $\eta \in T_x(X)$ and all $x \in X$.

Thus, by computing the evolution equation satisfied by holomorphic bisectional curvatures, the proof of Proposition (1.1) is reduced to an algebraic problem. In fact, we have

$$\frac{\partial}{\partial t} R_{\alpha\bar{\alpha}\zeta\bar{\zeta}} = \Delta R_{\alpha\bar{\alpha}\zeta\bar{\zeta}} + F(R)_{\alpha\bar{\alpha}\zeta\bar{\zeta}},$$

where F is an expression quadratic in the curvature tensor R. By applying the strong maximum principle to $u = R$, we are going to verify the null-vector condition for $F(R)$. In case of $n = 3$, condition (#) was verified in Bando [1] by using special coordinates. Our manipulation of the more complicated expression in arbitrary dimensions is motivated by similar considerations in the Einstein case as done in Mok-Zhong [19, Proposition (2.2.2)]. There, given a compact Kähler-Einstein manifold (X, g) of nonnegative bisectional curvature, we needed to show $\Delta R_{\alpha\bar{\alpha}\zeta\bar{\zeta}} = 0$ for certain zeros (α, ζ) of bisectional curvature. From standard commutation formulas we have

$$R_{\alpha\bar{\alpha}\zeta\bar{\zeta}} = \Delta R_{\alpha\bar{\alpha}\zeta\bar{\zeta}} + F_0(R)_{\alpha\bar{\alpha}\zeta\bar{\zeta}},$$

where the left-hand side denotes covariant derivatives of the Ricci tensor. Since (X, g) is Einstein, $R_{\alpha\bar{\alpha},\zeta\bar{\zeta}} = 0$ and we end up with

$$\Delta R_{\alpha\bar{\alpha}\zeta\bar{\zeta}} + F_0(R)_{\alpha\bar{\alpha}\zeta\bar{\zeta}} = 0.$$

Thus to show $\Delta R_{\alpha\bar{\alpha}\zeta\bar{\zeta}} = 0$ it suffices to show $F_0(R)_{\alpha\bar{\alpha}\zeta\bar{\zeta}} \geqslant 0$, just as in condition (#). In fact, in the general case (not necessarily Einstein) the evolution equation $\frac{\partial}{\partial t} g_{i\bar{j}} = -R_{i\bar{j}}$ essentially transforms the second order covariant derivatives $R_{\alpha\bar{\alpha},\zeta\bar{\zeta}}$ to time derivatives of bisectional curvatures, given by the formula (with $\{e_\mu\}$ an orthonormal basis)

$$\frac{\partial}{\partial t}R_{\alpha\bar{\alpha}\zeta\bar{\zeta}} = R_{\alpha\bar{\alpha},\zeta\bar{\zeta}} - \sum_{\mu} R_{\mu\bar{\alpha}}R_{\alpha\bar{\mu}\zeta\bar{\zeta}}.$$

Obviously, the zero order term vanishes whenever $R_{\alpha\bar{\alpha}\zeta\bar{\zeta}} = 0$ by the nonnegativity of bisectional curvatures. Thus the null-vector condition (#) for the two tensor expressions $F_0(R)$ and $F(R)$ are essentially the same. In Mok-Zhong [19], condition (#) for $F_0(R)$ was proved only for zeros (α, ζ) of bisectional curvatures such that α realizes the global maximum of holomorphic sectional curvature. For such zeros the special eigenspace decomposition of $T_x(X) = \mathbb{C}\alpha \oplus \mathscr{H}_\alpha \oplus \mathscr{N}_\alpha$ (cf. [19, Proposition 2.1]) significantly simplifies the linear algebra. As it turns out, we can generalize the method of [19] to arbitrary zeros of holomorphic bisectional curvatures.

(1.2) Proof of the null-vector condition (#). We fix $x \in X$ and choose an orthonormal basis $\{e_i\}$ of $T_x(X)$. Then

$$(1) \qquad \Delta R_{\alpha\bar{\alpha}\zeta\bar{\zeta}} = \tfrac{1}{2}\Big(\sum_i R_{\alpha\bar{\alpha}\zeta\bar{\zeta},i\bar{i}} + \sum_i R_{\alpha\bar{\alpha}\zeta\bar{\zeta},\bar{i}i}\Big),$$

where indices after a comma will always denote covariant differentiation. (The Laplacian is normalized so that Δ corresponds to $\frac{1}{2}\Delta$ in Mok-Zhong [19].) On the other hand

$$(2) \qquad R_{\alpha\bar{\alpha},\zeta\bar{\zeta}} = \sum_i R_{\alpha\bar{\alpha}i\bar{i},\zeta\bar{\zeta}}.$$

From standard commutation formulas we obtain

$$\begin{aligned}\Delta R_{\alpha\bar{\alpha}\zeta\bar{\zeta}} &- R_{\alpha\bar{\alpha},\zeta\bar{\zeta}} \\ &= \sum_{\mu,\bar{\nu}} |R_{\alpha\bar{\mu}\zeta\bar{\nu}}|^2 - \sum_{\mu,\nu} R_{\alpha\bar{\alpha}\mu\bar{\nu}} - \sum_{\mu,\nu} |R_{\alpha\bar{\zeta}\mu\bar{\nu}}|^2 \\ &\quad + \tfrac{1}{2}\sum R_{\alpha\bar{\mu}}R_{\mu\bar{\alpha}\zeta\bar{\zeta}} - \tfrac{1}{2}\sum R_{\mu\bar{\alpha}}R_{\alpha\bar{\mu}\zeta\bar{\zeta}} + \operatorname{Re} R_{\zeta\bar{\mu}}R_{\alpha\bar{\alpha}\mu\bar{\zeta}}.\end{aligned}$$

On the other hand, from the evolution equation

$$\frac{\partial}{\partial t}g_{i\bar{j}} = -R_{i\bar{j}}$$

and the equation for the curvature tensor

$$R_{i\bar{j}k\bar{l}} = -\frac{\partial^2 g_{i\bar{j}}}{\partial z_k \zeta \bar{z}_l} + \sum g^{\mu\bar{\nu}}\frac{\partial g_{i\bar{\nu}}}{\partial z_k}\frac{\partial g_{\mu\bar{j}}}{\partial \bar{z}_l}$$

in terms of local holomorphic coordinates, it is readily verified using complex geodesic coordinates (i.e. $g_{i\bar{j}} = \delta_{ij}$, $dg_{i\bar{j}} = 0$ at a fixed point $x \in X$) that we have

$$(4) \qquad \frac{\partial}{\partial t}R_{\alpha\bar{\alpha}\zeta\bar{\zeta}} = R_{\alpha\bar{\alpha},\zeta\bar{\zeta}} - \sum R_{\mu\bar{\alpha}}R_{\alpha\bar{\mu}\zeta\bar{\zeta}},$$

so that

$$\frac{\partial}{\partial t} R_{\alpha\bar{\alpha}\zeta\bar{\zeta}} = \Delta R_{\alpha\bar{\alpha}\zeta\bar{\zeta}} + F(R)_{\alpha\bar{\alpha}\zeta\bar{\zeta}}, \tag{5}$$

where

$$\begin{aligned} F(R)_{\alpha\bar{\alpha}\zeta\bar{\zeta}} = \sum_{\mu,\nu} R_{\alpha\bar{\alpha}\mu\bar{\nu}} R_{\nu\bar{\mu}\zeta\bar{\zeta}} - \sum_{\mu,\nu} \left| R_{\alpha\bar{\mu}\zeta\bar{\nu}} \right|^2 + \sum_{\mu,\nu} \left| R_{\alpha\bar{\zeta}\mu\bar{\nu}} \right|^2 \\ - \operatorname{Re}\left(R_{\alpha\bar{\mu}} R_{\mu\bar{\alpha}\zeta\bar{\zeta}} + R_{\zeta\bar{\mu}} R_{\alpha\bar{\alpha}\mu\bar{\zeta}} \right). \end{aligned} \tag{6}$$

Suppose now $(X, g_{i\bar{j}})$ is a compact Kähler manifold of nonnegative bisectional curvature and $R_{\alpha\bar{\alpha}\zeta\bar{\zeta}} = 0$ for a pair of nonzero vectors (α, ζ). To prove the null-vector condition (#) it suffices to show that

$$\sum_{\mu,\nu} R_{\alpha\bar{\alpha}\mu\bar{\nu}} R_{\nu\bar{\mu}\zeta\bar{\zeta}} \geqslant \sum_{\mu,\nu} \left| R_{\alpha\bar{\mu}\zeta\bar{\nu}} \right|^2, \tag{#$'$}$$

since the last term of $F(R)_{\alpha\bar{\alpha}\zeta\bar{\zeta}}$ vanishes by the inequality of Cauchy-Schwarz. Consider the Hermitian form $H_\alpha(\mu, \nu) = R_{\alpha\bar{\alpha}\mu\bar{\nu}}$ attached to α and let $\{e_\mu\}$ be an orthonormal basis of eigenvalues of H_α. In the basis we have

$$\sum_{\mu,\nu} R_{\alpha\bar{\alpha}\mu\bar{\nu}} R_{\nu\bar{\mu}\zeta\bar{\zeta}} = \sum_{\mu} R_{\alpha\bar{\alpha}\mu\bar{\mu}} R_{\mu\bar{\mu}\zeta\bar{\zeta}}. \tag{7}$$

From now on we will always use such an orthonormal basis and show

$$\sum_{\mu} R_{\alpha\bar{\alpha}\mu\bar{\mu}} R_{\mu\bar{\mu}\zeta\bar{\zeta}} \geqslant \sum_{\mu,\nu} \left| R_{\alpha\bar{\mu}\zeta\bar{\nu}} \right|^2. \tag{#$''$}$$

As in Mok-Zhong [19, second proof of Proposition (2.2.2)] we consider the function

$$G_\chi(\varepsilon) = R\left(\alpha + \varepsilon\chi,\ \overline{\alpha + \varepsilon\chi},\ \zeta + \varepsilon \sum_{\mu} C_\mu e_\mu,\ \overline{\zeta + \varepsilon \sum_{\mu} C_\mu e_\mu} \right). \tag{8}$$

We have $G_\chi(\varepsilon) \geqslant 0$ so that $\partial^2 G_\chi(0)/\partial\varepsilon^2 \geqslant 0$, while

$$\begin{aligned} \frac{\partial^2 G_\chi}{\partial \varepsilon^2}(0) = R_{\chi\bar{\chi}\zeta\bar{\zeta}} + \sum_{\mu} \left| C_\mu \right|^2 R_{\alpha\bar{\alpha}\mu\bar{\mu}} + 2 \operatorname{Re} \sum_{\mu} C_\mu R_{\alpha\bar{\chi}\zeta\bar{\mu}} \\ + 2 \operatorname{Re} \sum_{\mu} C_\mu R_{\alpha\bar{\zeta}\mu\bar{\chi}}. \end{aligned} \tag{9}$$

Consider the quadratic form

$$\begin{aligned} Q((z_0, \cdots, z_n); (z_0, \cdots, z_n)) = \left| z_0 \right|^2 R_{\chi\bar{\chi}\zeta\bar{\zeta}} + \sum_{1 \leqslant \mu \leqslant n} \left| z_\mu \right|^2 R_{\alpha\bar{\alpha}\mu\bar{\mu}} \\ + 2 \operatorname{Re} \sum_{1 \leqslant \mu \leqslant n} \left(z_0 \bar{z}_\mu R_{\alpha\bar{\chi}\zeta\bar{\mu}} + z_0 z_\mu R_{\alpha\bar{\zeta}\mu\bar{\chi}} \right). \end{aligned} \tag{10}$$

The variational inequality $\partial^2 G_\chi(0)/\partial\varepsilon^2 \geqslant 0$ implies that Q is nonnegative for z_0 real. In fact, by a suitable scaling $\alpha \mapsto e^{i\varnothing}\alpha$, $\zeta \mapsto e^{i\psi}\zeta$, it follows easily that Q defines a positive semidefinite Hermitian symmetric form. We will only consider $z_0 = x_0$ real. For $\theta = (\theta_1, \cdots, \theta_n)$ consider the real symmetric bilinear form defined by

$$\begin{aligned} Q_\theta&((x_0,\cdots,x_n);(x_0,\cdots,x_n)) \\ &= Q\big((x_0, x_1e^{i\theta_1},\cdots,x_ne^{i\theta_n});(x_0,x_1e^{i\theta_1},\cdots,x_ne^{i\theta_n})\big) \\ &= |x_0|^2 R_{\chi\bar\chi\zeta\bar\zeta} + \sum_{1\leqslant\mu\leqslant n}|x_\mu|^2 R_{\alpha\bar\alpha\mu\bar\mu} \\ &\quad + \sum_{1\leqslant\mu\leqslant n} x_0x_\mu\big(\operatorname{Re} e^{-i\theta_\mu}R_{\alpha\bar\chi\zeta\bar\mu} + \operatorname{Re} e^{i\theta_\mu}R_{\alpha\bar\zeta\mu\bar\chi}\big). \end{aligned} \tag{11}$$

Since Q_θ is positive semidefinite, the matrix

$$\begin{bmatrix} R_{\chi\bar\chi\zeta\bar\zeta} & S_1 & \cdots & S_n \\ S_1 & R_{\alpha\bar\alpha 1\bar 1} & & \\ \vdots & & \ddots & 0 \\ S_n & 0 & & R_{\alpha\bar\alpha n\bar n} \end{bmatrix}$$

with $S_\mu = \operatorname{Re}(e^{-i\theta_\mu}R_{\alpha\bar\chi\zeta\bar\mu} + e^{i\theta_\mu}R_{\alpha\bar\zeta\mu\bar\chi})$ is positive semidefinite. Arrange the basis $\{e_\mu\}$ so that $R_{\alpha\bar\alpha l\bar l} \geqslant R_{\alpha\bar\alpha 2\bar 2} \geqslant \cdots \geqslant R_{\alpha\bar\alpha m\bar m} > 0$ and $R_{\alpha\bar\alpha l\bar l} = 0$ for $l > m$. By considering the submatrix obtained by replacing n by m and by computing the determinant we obtain the inequality

$$\begin{aligned} R_{\chi\bar\chi\zeta\bar\zeta}&R_{\alpha\bar\alpha 1\bar 1}\cdots R_{\alpha\bar\alpha m\bar m} \\ &\geqslant \sum_{1\leqslant\mu\leqslant m}|S_\mu|^2 R_{\alpha\bar\alpha 1\bar 1}\cdots \hat R_{\alpha\bar\alpha\mu\bar\mu}\cdots R_{\alpha\bar\alpha m\bar m}, \end{aligned} \tag{12}$$

where the symbol ^ denotes omission. Thus

$$R_{\chi\bar\chi\zeta\bar\zeta} \geqslant \sum_{1\leqslant\mu\leqslant m}\frac{|S_\mu|^2}{R_{\alpha\bar\alpha\mu\bar\mu}}. \tag{13}$$

We assert that in the inequality in (13) $|S_\mu|^2$ can be replaced by the quantity $|R_{\alpha\bar\zeta\mu\bar\chi}|^2 + |R_{\alpha\bar\chi\zeta\bar\mu}|^2$. To prove this we write $A_\mu = R_{\alpha\bar\zeta\mu\bar\chi}$, $B_\mu = R_{\alpha\bar\chi\zeta\bar\mu}$. Then by (13) we have, for any $\theta = (\theta_1, \cdots, \theta_m)$ real,

$$R_{\chi\bar\chi\zeta\bar\zeta} \geqslant \sum_{1\leqslant\mu\leqslant k}\frac{1}{R_{\alpha\bar\alpha\mu\bar\mu}}\left|\frac{1}{2}e^{i\theta_\mu}(A_\mu + \bar B_\mu) + \frac{1}{2}e^{-i\theta_\mu}(\bar A_\mu + B_\mu)\right|^2. \tag{14}$$

By choosing θ_μ such that $e^{i\theta_\mu}(A_\mu + \bar B_\mu)$ is real we have

$$R_{\chi\bar\chi\zeta\bar\zeta} \geqslant \sum_{1\leqslant\mu\leqslant k}\frac{1}{R_{\alpha\bar\alpha\mu\bar\mu}}|A_\mu + \bar B_\mu|^2. \tag{15}$$

Moreover, the inequality remains valid when α is replaced by $e^{i\theta}\alpha$. In this case $A_\mu = R_{\alpha\bar{\xi}\mu\bar{\chi}}$ is replaced by $e^{i\phi}A_\mu$ and $\overline{B}_\mu = \overline{R}_{\alpha\bar{\chi}\zeta\bar{\mu}}$ is replaced by $e^{-i\phi}\overline{B_\mu}$, so that we have the general inequality

$$(16)\qquad R_{\chi\bar{\chi}\zeta\bar{\zeta}} \geqslant \sum_{1\leqslant\mu\leqslant k} \frac{1}{R_{\alpha\bar{\alpha}\mu\bar{\mu}}}\left|e^{i\phi}A_\mu + e^{-i\phi}\overline{B_\mu}\right|^2.$$

In order to establish the inequality

$$(17)\qquad R_{\chi\bar{\chi}\zeta\bar{\zeta}} \geqslant \sum_{1\leqslant\mu\leqslant k} \frac{1}{R_{\alpha\bar{\alpha}\mu\bar{\mu}}}\left(\left|R_{\alpha\bar{\chi}\zeta\bar{\mu}}\right|^2 + \left|R_{\alpha\bar{\xi}\chi\bar{\mu}}\right|^2\right),$$

it suffices therefore to show the following

Lemma. *For any two complex numbers σ and τ*

$$\frac{1}{2\pi}\int_0^{2\pi} \left|e^{i\phi}\sigma + e^{-i\phi}\tau\right|^2 d\phi = |\sigma|^2 + |\tau|^2.$$

Proof of Lemma. Clearly the integral equals

$$\frac{1}{4\pi}\int_0^{2\pi}\left(\left|e^{i\phi}\sigma + e^{-\phi}\tau\right|^2 + \left|e^{i(\phi+\pi/2)}\sigma + e^{-i(\phi+\pi/2)}\tau\right|^2\right) d\phi$$

$$= \frac{1}{4\pi}\int_0^{2\pi}\left(\left|e^{i\phi}\sigma + e^{-\phi}\tau\right|^2 + \left|ie^{i\phi}\sigma - ie^{-\phi}\tau\right|^2 d\phi.\right.$$

But now

$$\left|e^{i\phi}\sigma + e^{-i\phi}\tau\right|^2 = |\sigma|^2 + |\tau|^2 + 2\,\mathrm{Re}\, e^{2i\phi}\sigma\bar{\tau},$$

$$\left|e^{i\phi}\sigma - e^{-i\phi}\tau\right|^2 = |\sigma|^2 + |\tau|^2 - 2\,\mathrm{Re}\, e^{2i\phi}\sigma\bar{\tau}.$$

Thus,

$$\frac{1}{2\pi}\int_0^{2\pi}\left|e^{i\phi} + e^{-i\phi}\tau\right|^2 d\phi = \frac{1}{4\pi}\int_0^{2\pi} 2\left(|\sigma|^2 + |\tau|^2\right) d\phi = |\sigma|^2 + |\tau|^2.$$

Having proved (17), we rewrite it in the form

$$(18)\qquad R_{\alpha\bar{\alpha}\chi\bar{\chi}}R_{\chi\bar{\chi}\zeta\bar{\zeta}} \geqslant \sum_{1\leqslant\mu\leqslant m} \frac{R_{\alpha\bar{\alpha}\chi\bar{\chi}}}{R_{\alpha\bar{\alpha}\mu\bar{\mu}}}\left(\left|R_{\alpha\bar{\chi}\zeta\bar{\mu}}\right|^2 + \left|R_{\alpha\bar{\xi}\chi\bar{\mu}}\right|^2\right).$$

Now we take $\chi = e_1, \cdots, e_m$ and add to (18) the inequality obtained by interchanging the roles of χ and μ, obtaining by summation

$$(19)\qquad 2\sum_{1\leqslant\mu\leqslant m} R_{\alpha\bar{\alpha}\mu\bar{\mu}}R_{\mu\bar{\mu}\zeta\bar{\zeta}} \geqslant \sum_{1\leqslant\mu,\chi\leqslant m}\left(\frac{R_{\alpha\bar{\alpha}\chi\bar{\chi}}}{R_{\alpha\bar{\alpha}\mu\bar{\mu}}} + \frac{R_{\alpha\bar{\alpha}\mu\bar{\mu}}}{R_{\alpha\bar{\alpha}\chi\bar{\chi}}}\right)\left|R_{\alpha\chi\zeta\bar{\mu}}\right|^2.$$

But, for any two positive real numbers a and b we have

$$\frac{a}{b} + \frac{b}{a} - 2 = \frac{a^2 + b^2 - 2ab}{ab} \geqslant 0,$$

so that $\frac{1}{2}(a/b + b/a) \geqslant 1$, thus giving

$$(20) \qquad \sum_{1 \leqslant \mu \leqslant m} R_{\alpha\bar{\alpha}\mu\bar{\mu}} R_{\mu\bar{\mu}\zeta\bar{\zeta}} \geqslant \sum_{1 \leqslant \mu \leqslant m} \left| R_{\alpha\bar{\chi}\zeta\bar{\mu}} \right|^2 .$$

Moreover, for $\mu > m$ we have $R_{\alpha\bar{\alpha}\mu\bar{\mu}} = 0$ and $R_{\alpha\bar{\chi}\zeta\bar{\mu}} = 0$. In fact, for any θ such that $R_{\alpha\bar{\alpha}\theta\bar{\theta}} = 0$ (we write $\theta \in \mathcal{N}_\alpha$) we have $R_{\alpha\bar{\chi}\theta\bar{\theta}} = 0$ by the Cauchy-Schwarz inequality, so that by polarization $R_{\alpha\bar{\chi}\zeta\bar{\mu}} = 0$ for $\zeta \in \mathcal{N}_\alpha$, $\mu > m$. From this we obtain finally the inequality

$$(\#)'' \qquad \sum_{\mu} R_{\alpha\bar{\alpha}\mu\bar{\mu}} R_{\mu\bar{\mu}\zeta\bar{\zeta}} \geqslant \sum_{\mu\nu} \left| R_{\alpha\bar{\mu}\zeta\bar{\nu}} \right|^2$$

for an eigenbasis $\{e_\mu\}$ of the $\mathscr{H}_\alpha$ and thus the inequality

$$(\#)' \qquad \sum_{\mu,\nu} R_{\alpha\bar{\alpha}\mu\bar{\nu}} R_{\nu\bar{\mu}\zeta\bar{\zeta}} \geqslant \sum_{\mu,\nu} \left| R_{\alpha\bar{\mu}\zeta\bar{\nu}} \right|^2$$

for any orthonormal basis $\{e_\mu\}$, proving $(\#)$ and hence the fact that evolved metrics $g_{i\bar{j}}(t)$ carry nonnegative holomorphic bisectional curvature. By considering the evolution of Ricci curvatures it follows easily that $(X, g_{i\bar{j}}(t))$ carries positive Ricci curvature if $(X, g_{i\bar{j}})$ is of positive Ricci curvature at one point (cf. Bando [1]). Moreover, if $R_{\alpha\bar{\alpha}\alpha\bar{\alpha}} = 0$ for an evolved metric, then we have

$$(21) \qquad \frac{\partial}{\partial t} R_{\alpha\bar{\alpha}\alpha\bar{\alpha}} = 0, \qquad \Delta R_{\alpha\bar{\alpha}\alpha\bar{\alpha}} = F(R)_{\alpha\bar{\alpha}\alpha\bar{\alpha}} = 0.$$

From the proof of Proposition (1.2) we know that, for an evolved metric, $R_{\alpha\bar{\zeta}\mu\bar{\nu}} = 0$ whenever $R_{\alpha\bar{\alpha}\zeta\bar{\zeta}} = 0$. (We proved that $F(R)_{\alpha\bar{\alpha}\zeta\bar{\zeta}} \geqslant \sum_{\mu,\nu} |R_{\alpha\bar{\zeta}\mu\bar{\nu}}|^2$.) Thus, $R_{\alpha\bar{\alpha}\alpha\bar{\alpha}} = 0$ implies $R_{\alpha\bar{\alpha}\mu\bar{\nu}} = 0$ and in particular $R_{\alpha\bar{\alpha}} = 0$, contradicting the positivity of the Ricci tensor. With this remark we have finished the proof of Proposition (1.1).

2. Rational curves of minimal degree and the variety $\mathscr{S} \subset \mathbb{P}T_X$

(2.1) In order to prove Theorem 1 and hence our Main Theorem, we can consider without loss of generality simply-connected compact Kähler manifolds (X, h) with $b_2(X) = 1$ and $h = g_{\alpha\bar{\zeta}}(t)$ for some $t > 0$ as in §1. Thus by Proposition (1.2) we may assume that (X, h) carries nonnegative holomorphic bisectional curvature, positive Ricci curvature, and positive holomorphic sectional curvature. Our purpose is to show that either X is biholomorphic to $\mathbb{P}^n$, or the holonomy group at $P \in X$ does not act transitively on the unit sphere of $T_P(X)$. In the latter case by Berger's characterization of locally symmetric spaces we have either (i) (X, h) is an irreducible compact Hermitian symmetric space of rank $\geqslant 2$ or (ii) (X, h) is globally reducible as a Kähler manifold. Observing that the second possibility contradicts the condition $b_2(X) = 1$, we obtain then a proof of Theorem 1 and hence the Main Theorem.

We will need the following theorem of Mori.

Theorem (*Mori* [20]). *Let X be an n-dimensional projective algebraic manifold with ample anticanonical line bundle K_X^{-1}. Then there exists a rational curve f_0: $\mathbb{P}^1 \to X$ such that $f_0^*(K_X^{-1}) \cong \mathcal{O}(q)$ for some q satisfying $0 < q \leqslant n + 1$. Let $\mathscr{C}$ be a connected component of the Chow variety of rational cycles $[f(\mathbb{P}^1)]$ such that $f^*(K_X^{-1}) = \mathcal{O}(q)$ with minimal q. Denote by T_X the holomorphic tangent bundle of X. Suppose $q = n + 1$, and there is a point $x \in X$, such that $f^*(T_X)$ is ample for all $[f(\mathbb{P}^1)] \in \mathscr{C}$, $f(\mathbb{P}^1)$ passing through x. Then X is biregular to the projective space $\mathbb{P}^n$.*

Mori's theorem is valid over any algebraically closed field k. In case $k = \mathbb{C}$ the last statement implies that X is biholomorphic to $\mathbb{P}^n$. Our statement of Mori's theorem is a slight variation of the original. It can be deduced immediately from his proof.

Since our Kähler manifold (X, h) carries positive Ricci curvature, K_X^{-1} is ample by the Kodaira vanishing theorem. In particular X is projective algebraic. Hence, Mori's result applies and we obtain a nontrivial f_0: $\mathbb{P}^1 \to X$. We will assume $f_0^*(K_X^{-1}) \cong \mathcal{O}(q)$ with q minimal and we call such an f_0 a rational curve of minimal degree. We are going to define a complex-analytic subvariety $\mathscr{S} \subset \mathbb{P}T_X$ by deforming some rational curve of minimal degree. The construction of $\mathscr{S}$ in the two cases (i) $q < n + 1$ and (ii) $q = n + 1$ but $X \not\cong \mathbb{P}^n$ will be different.

We first fix some notations and state a number of well-known preparatory theorems. For any compact complex manifold W and any two complex analytic subspaces $Y, Z \subset W$ of complementary dimensions intersecting properly, we will denote by $Y \cdot Z$ the intersection multiplicity of Y and Z. This definition extends to rational combination of irreducible subspaces in an obvious way. Let L be a positive line bundle over W and C be a 1-dimensional cycle of W, i.e., a positive integral combination of irreducible curves. For some positive integer m the line bundle L^m admits a nontrivial section s which does not vanish on any irreducible component of C. Writing $Z(s)$ for the zero-divisor of s, we define $L \cdot C$ to be $\frac{1}{m}Z(s) \cdot C$, which is clearly independent of the choice of s. If C is an irreducible rational curve and f: $\mathbb{P}^1 \mapsto C$ is a normalization of C, then clearly $L \cdot C = p$ if $f^*L \cong \mathcal{O}(p)$ over $\mathbb{P}^1$.

We consider a holomorphic map f_0: $\mathbb{P}^1 \mapsto X$, where X carries a Kähler metric h as described above and f_0 is of minimal degree. It follows that f_0: $\mathbb{P}^1 \mapsto f_0(\mathbb{P}^1)$ is injective generically so that it is nothing but a normalization of the image curve $f_0(\mathbb{P}^1)$. By the Kodaira Vanishing Theorem X is projective algebraic. Let $\mathrm{Chow}(K_X^{-1}; q)$ be the Chow variety of the 1-dimensional cycle C such that $K_X^{-1} \cdot C = q$. As is well known (Chow-van der Waerden [6]), it is a projective variety. Let $\mathscr{C} \subset \mathrm{Chow}(K_X^{-1}; q)$ be the connected component of $\mathrm{Chow}(K_X^{-1}; q)$ containing the reduced irreducible curve $[f_0(\mathbb{P}^1)]$. Here we denote by $[Y]$ the cycle defined by a pure-dimensional complex subspace Y (possibly not reduced) of X. On the other hand consider the nonsingular projective variety $\mathbb{P}^1 \times X$ and the nonsingular projective curve $\mathrm{Graph}(f_0) \subset \mathbb{P}^1 \times X$. Write $\mathrm{Chow}(K_{\mathbb{P}^1 \times X}^{-1}; q+2)$ for the Chow variety of 1-dimensional cycles D such that $K_{\mathbb{P}^1 \times X}^{-1} \cdot D = q + 2$. Consider any holomorphic map f: $\mathbb{P}^1 \mapsto X$ such that $f^*(K_X^{-1}) \cong \mathcal{O}(q)$. Then, $[\mathrm{Graph}(f)] \in \mathrm{Chow}(K_{\mathbb{P}^1 \times X}^{-1}; q+2)$ since $T_{\mathbb{P}^1} \cong \mathcal{O}(2)$. Let $\mathscr{H} \subset \mathrm{Chow}(K_{\mathbb{P}^1 \times X}^{-1}; q+2)$ be the Zariski open subset consisting of graphs of holomorphic maps f: $\mathbb{P}^1 \mapsto X$ such that $[f(\mathbb{P}^1)] \in \mathscr{C}$. Write $\mathscr{H} = \mathscr{H}_1 \cup \cdots \cup \mathscr{H}_m$ for the decomposition of $\mathscr{H}$ into irreducible components. The projection map $\mathbb{P}^1 \times X \mapsto X$ induces a natural holomorphic map Φ: $\mathscr{H} \mapsto \mathrm{Chow}(K_X^{-1}; q)$.

The automorphism group $\mathrm{Aut}(\mathbb{P}^1)$ of $\mathbb{P}^1$ acts on $\mathscr{H}$ on the right by composition. Writing $[f]$ for $[\mathrm{Graph}(f)]$ whenever $[\mathrm{Graph}(f)] \in \mathscr{H}$, the $\mathrm{Aut}(\mathbb{P}^1)$-action on $\mathscr{H}$ is given by $\varphi([f]) = [f \circ \varphi]$. Since $f = \mathbb{P}^1 \mapsto f(\mathbb{P}^1)$ is a normalization of $f(\mathbb{P}^1) \subset X$, it follows readily that the action is free. Regarding free actions of complex Lie groups on complex manifolds, we have

Theorem A (*Holmann* [10], *Kaup* [13]). *Let W be a complex manifold and G be a complex Lie group acting freely on W (i.e. without fixed points). Write W/G for the set of equivalence classes of W modulo the equivalence relation identifying points in the same G-orbit. Then, one can equip W/G with a (unique) natural structure of a normal complex space such that the natural map $W \to W/G$ is holomorphic.*

More precisely, one endows W/G with the structure of a local ringed space by equipping it with the structure sheaf $\mathcal{O}_{W/G}$ arising from G-invariant holomorphic functions defined on certain subsets of W, and show that the ringed space $(W/G, \mathcal{O}_{W/G})$ is isomorphic as a ringed space to a normal complex space. (From now on W/G will be understood to carry this complex structure.) This is done by using holomorphic "Quasischnitte" (quasi-sections) of W modulo G in Kaup [13]. Theorem A as it stands will be sufficient for us. However, in order to streamline the presentation, we will use the following criterion on the nonsingularity of W/G. We use the same notations and hypothesis as in Theorem A.

Theorem A′. *Suppose for each $w \in W$ there exists a local complex submanifold S_w such that $w \in S$, $\dim_{\mathbb{C}} S_w = \dim_{\mathbb{C}} W - \dim_{\mathbb{C}} G$, S_w is transverse to the G-orbit Gw at w, and the natural map σ: $S_w \mapsto W/G$ is injective. Then the unique natural normal complex structure on W/G is nonsingular. Moreover, the* *natural map $W \mapsto W/G$ is holomorphic and realizes W as a principal holomorphic G-bundle over W/G.*

The proof of Theorem A′ is in fact very elementary (independent of Theorem A). In terms of the terminology of Kaup [13] it means that the (holomorphic) quasi-sections of W modulo G are actually local (holomorphic) sections. In this case, the sets $\sigma(S_w) \subset W/G$ can be used as a covering of W/G by open submanifolds in an obvious way to yield the nonsingular complex structure on W/G. The second statement is then an immediate consequence.

We are interested in the deformation of [Graph(f_0)] in $\mathbb{P}^1 \times X$. Regarding the deformation theory of nonsingular subvarieties we have

Theorem B (*Kodaira* [14, *Theorem* 2]). *Let Z be a complex manifold and $Y \subset Z$ a connected compact complex submanifold. Let $N_{Y|Z}$ denote the normal bundle of Y in Z. Suppose $H^1(Y, N_{Y|Z}) = 0$. Then there exists an open set $\Omega \subset \mathbb{C}^m$ for some m, $0 \in \Omega$, and a parametrized family π: $\mathcal{Y} \mapsto \Omega$ of submanifolds $Y_t = \pi^{-1}(t) \subset \{t\} \times Z$, $\mathcal{Y} \subset \Omega \times Z$, with $Y_0 = \{0\} \times Y$, such that the corresponding germ of deformation spaces of compact complex submanifolds of Z is a versal family at* 0. *Moreover* $m = \dim H^0(Y, N_{Y|Z})$.

If one considers deformation families of compact complex submanifolds centered at $Y_0 = \{0\} \times Y$ such that every fiber corresponds to a submanifold of Z containing a distinguished submanifold S of Y, one obtains an obvious relative version of the above theorem of Kodaira, namely, one replaces $H^0(Y, N_{Y|Z})$ by $H^0(Y, N_{Y|Z} \otimes \mathcal{I}_S)$ and requires the vanishing of $H^1(Y, N_{Y|Z} \otimes \mathcal{I}_S)$, where $\mathcal{I}_S$ stands for the ideal sheaf of S in Y. In general, one does not need the vanishing of $H^1(Y, N_{Y|Z} \otimes \mathcal{I}_S)$ for the existence of versal families of deformations of submanifolds. (The absolute case corresponds to $S = \varnothing$.) The stated versions will however be sufficient for us.

We will need to analyze families of rational curves in X. For this we need the following classical result of Enriques-Noether.

Theorem C (*Enriques-Noether*; *cf*., *e.g*., *Barth-Peters-Van de Ven* [2] *or Beauville* [3]). *Let S be a nonsingular compact complex manifold and φ: $S \mapsto C$ be a nontrivial holomorphic map of S onto a compact Riemann surface C such that for some $c \in C$, $\varphi^{-1}(c)$ is a nonsingular rational curve. Then, there exists a minimal model S_0 of S such that S is obtained from S_0 by a finite number of quadratic transforms* (*Hopf blow-ups*) *and such that φ: $S \mapsto C$ induces naturally a map φ_0: $S_0 \mapsto C$ realizing S_0 as a ruled surface over C.*

We also record the following well-known criterion for blowing down nonsingular curves Γ in a surface Σ, due to Mumford [15] and Grauert [7].

Theorem D. *Let Σ be a complex surface, not necessarily compact, and let $\Gamma \subset \Sigma$ be a nonsingular compact complex curve. Then, there exists a proper* *holomorphic map* $\chi\colon V \mapsto \mathbb{C}^2$ *of maximal rank of some open neighborhood* V *of* Γ *in* Σ *such that* $\chi(\Gamma) = 0$ *if and only if the normal bundle* $N_{\Gamma|\Sigma}$ *of* Γ *in* Σ *is negative.*

Concerning families of vector bundles we need the following special case of the Direct Image Theorem and the Semi-Continuity Theorem of Grauert.

Theorem E (*special case of Grauert* [7]). *Let* Z *be a compact complex space,* $\Omega \subset \mathbb{C}^N$ *an open set, and* $V \mapsto \Omega \times Z$ *a holomorphic vector bundle. Consider the proper projection map* $\pi\colon \Omega \times Z \to \Omega$. *Then, the zeroth direct image* $\pi_*(\mathscr{V})$ (*of the locally free sheaf* $\mathscr{V}$ *over* $\Omega \times Z$ *defined by* V) *is a coherent sheaf on* Ω. *Moreover, if* $\pi_*(\mathscr{V})$ *is locally free and of rank* r *at a generic point* $w \in \Omega$, *then* $\dim H^0(\{w\} \times Z, V) \geqslant r$ *for all* $w \in \Omega$ *and strict inequality holds if and only if* w *belongs to some analytic subvariety* E *of* Ω.

(2.2) Deformation of rational curves of minimal degree. We use the notations as given in (2.1). Recall that $f_0\colon \mathbb{P}^1 \mapsto X$ is a given rational curve of minimal degree, $\mathscr{C} \subset \mathrm{Chow}(K_x^{-1}, q)$ is a connected projective variety containing $[f_0(\mathbb{P}^1)]$, and $\mathscr{H} \subset \mathrm{Chow}(K_{\mathbb{P}^1\times X}^{-1}, q+2)$ is a quasi-projective variety consisting of cycles in $\mathrm{Chow}(K_{\mathbb{P}^1\times X}^{-1}, q+2)$ representing graphs of $f\colon \mathbb{P}^1 \mapsto X$. We assert

Proposition (2.2.1). $\mathscr{H}$ *is a nonsingular* $(q+n)$*-dimensional quasi-projective variety.*

Proof. Let $[f] \in \mathscr{H}$ and G be its graph ($[f] = [\mathrm{Graph}(f)]$ for short). Since G is by definition nonsingular the tangent space of $\mathscr{H}$ at $[f]$ can be identified with the space of infinitesimal deformations of G in $\mathbb{P}^1 \times X$ as a complex submanifold. Let $N_{G|\mathbb{P}^1\times X}$ denote the normal bundle of G in $\mathbb{P}^1 \times X$. Then

$$H^1(G, N_{G|\mathbb{P}^1\times X}) = H^1(G, T_{\mathbb{P}^1\times X}/T_G) \cong H^1\big(\mathbb{P}^1, (T_{\mathbb{P}^1} \oplus f^*T_X)/\sigma_* T_{\mathbb{P}^1}\big),$$

where $\sigma\colon T_{\mathbb{P}^1} \mapsto T_{\mathbb{P}^1} \oplus f^*T_X$ is the bundle map over $\mathbb{P}^1$ induced by the identity map in the first component and $df\colon T_{\mathbb{P}^1} \mapsto T_X$ in the second component. Obviously $\sigma_* T_{\mathbb{P}^1}$ is a line subbundle of $T_{\mathbb{P}^1} \oplus f^*T_X$ and $(T_{\mathbb{P}^1} \oplus f^*T_X)/\sigma_* T_{\mathbb{P}^1} \cong f^*T_X$ naturally. Thus,

$$H^1(G, N_{G|\mathbb{P}^1\times X}) \cong H^1(\mathbb{P}^1, f^*T_X).$$

Since (X, h) carries nonnegative holomorphic bisectional curvature, in the splitting $f^*T_X \cong \mathscr{O}(a_1) \oplus \cdots \oplus \mathscr{O}(a_n)$, $a_1 \geqslant \cdots \geqslant a_n$, into holomorphic line bundles we must have $a_n \geqslant 0$. It follows therefore that $H^1(G, N_{G|\mathbb{P}^1\times X}) = 0$ so that by Theorem B of Grauert $\mathscr{H}$ is smooth at $[f]$. Moreover,

$$\begin{aligned}\dim_{[f]} \mathscr{H} &= \dim H^0(G, N_{G|\mathbb{P}^1\times X}) \\ &= \dim H^0(\mathbb{P}^1, f^*T_X) = \sum_{i=1}^{n} a_i + 1 = q + n.\end{aligned}$$

The proof of Proposition (2.2.1) is complete.

Recall that $\mathrm{Aut}(\mathbb{P}^1)$ acts on $\mathscr{H}$ by composition on the right. Recall that $\mathscr{H}_1, \cdots, \mathscr{H}_m$ are the connected components of $\mathscr{H}$ that are mapped into $\mathscr{C}$. A priori $\mathscr{H}_i/\mathrm{Aut}(\mathbb{P}^1)$, $1 \leqslant i \leqslant m$, may be noncompact. We are going to prove that $m = 1$, that Φ: $H \to C$ is surjective, and that $\mathscr{H}/\mathrm{Aut}(\mathbb{P}^1)$ can be identified with a normalization $\hat{\mathscr{C}}$ of $\mathscr{C}$. Let π: $\mathscr{E} \mapsto \mathscr{C}$ be the family of cycles parametrized by $\mathscr{C}$. Namely $\mathscr{E} \subset \mathscr{C} \times X$ is the subspace defined by $\mathscr{E} = \{([C], x) \in \mathscr{C} \times X\colon x \in C\}$. Since each irreducible component of $\mathscr{C}$ contains the reduced irreducible rational curve $[f_0(\mathbb{P}^1)]$ it is clear that the generic fiber of π: $\mathscr{E} \mapsto \mathscr{C}$ is a (possibly singular) reduced irreducible rational curve. Moreover, by passing to the normalization $\hat{\mathscr{E}}$, $\hat{\mathscr{C}}$ and $\hat{\pi}$: $\hat{\mathscr{E}} \mapsto \hat{\mathscr{C}}$ and computing the arithmetic genus, it is well known that every fiber of $\hat{\pi}$: $\hat{\mathscr{E}} \mapsto \hat{\mathscr{C}}$ (or of π: $\mathscr{E} \mapsto \mathscr{C}$) is a finite union of rational curves. By the minimality of q one sees easily that each fiber must be irreducible and reduced. Write $\mathscr{Q}_i = \mathscr{H}_i/\mathrm{Aut}(\mathbb{P}^1)$, $1 \leqslant i \leqslant m$. Then, the natural map Φ: $\mathscr{H} \mapsto \mathrm{Chow}(K^{-1}_{\mathbb{P}^1 \otimes X}; q + 2)$ induces maps ν_i: $Q_i \mapsto \mathscr{C}$. Since each fiber of π: $\mathscr{E} \mapsto \mathscr{C}$ is a reduced irreducible rational curve, it follows from the definition of $\mathscr{H}_i$, $1 \leqslant i \leqslant m$, that $\bigcup_{1 \leqslant i \leqslant m} \nu_i(\mathscr{Q}_i) = \mathscr{C}$. We assert that the action of $\mathrm{Aut}(\mathbb{P}^1)$ on $\mathscr{H}_i$ satisfies the criterion in Theorem A′ so that $\mathscr{Q}_i$ is a complex manifold. Namely, we have

Lemma. *Let $1 \leqslant i \leqslant m$ and $[f] \in \mathscr{H}_i$. Then there exists a smooth $(q + n - 3)$-dimensional holomorphic family S of rational curves $[g] \in \mathscr{H}_i$, S containing $[f]$, such that any two $[g_1]$, $[g_2] \in S$ represent distinct cycles $[g_1(\mathbb{P}^1)]$, $[g_2(\mathbb{P}^1)] \subset X$.*

Proof of Lemma. Recall that $\mathscr{H}_i$ is smooth at $[f]$ and that $\dim_{\mathbb{C}} \mathscr{H}_i = q + n$. Consider the graph G of f, $G \subset \mathbb{P}^1 \times X$. We have the identification $T_{[f]}(\mathscr{H}_i) \cong H^0(G, N_{G|\mathbb{P}^1 \times X})$. As above we identify $H^0(G, N_{G|\mathbb{P}^1 \times X})$ with $H^0(\mathbb{P}^1, f^*T_X)$. The map f_*: $T_{\mathbb{P}^1} \to T_X$ induced by f is nontrivial and injective except at possibly a finite number of points. It induces an injection τ_f: $\mathbb{C}^3 \cong H^0(\mathbb{P}^1, T_{\mathbb{P}^1}) \to H^0(\mathbb{P}^1, f^*T_X)$. Now let $s_1, \cdots, s_{q+n-3}$ be $q + n - 3$ holomorphic sections of f^*T_X over $\mathbb{P}^1$ which are $\mathbb{C}$-linearly independent in $H^0(\mathbb{P}^1, f^*T_X)/\tau_f(H^0(\mathbb{P}^1, T_{\mathbb{P}^1}))$. Let S be a local $(q + n - 3)$-dimensional nonsingular holomorphic family of rational curves $[g] \in \mathscr{H}_1$ such that $[f] \in S$ and $T_{[f]}(S) = \sum_{i=1}^{q+n-3} \mathbb{C} s_i$. We assert that for S sufficiently small the natural mapping σ: $S \mapsto \mathscr{H}_i/\mathrm{Aut}(\mathbb{P}^1)$ is injective. For any $[g] \in S$ let τ_g: $(\mathbb{P}^1, T_{\mathbb{P}^1})$ $(\cong \mathbb{C}^3) \to H^0(\mathbb{P}^1, g^*T_X) \cong T_{[g]}(\mathscr{H}_1)$ be induced by g. The action of $\mathrm{Aut}(\mathbb{P}^1)$ on $\mathscr{H}_i$ defines a holomorphic foliation of $\mathscr{H}_i$ by 3-dimensional closed submanifolds $L_{[g]} = \{[g \circ \varphi]\colon \varphi \in \mathrm{Aut}(\mathbb{P}^1)\}$ which are integral submanifolds of the distribution $g \mapsto \tau_g(H^0(\mathbb{P}^1, T_{\mathbb{P}^1}))$. Parametrize S by an open neighborhood U of 0 in $\mathbb{C}^{q+n-3}$ and write $S = \{[f_t]\colon t \in U\}$. There exists an open neighborhood $\mathscr{U}$ of $[f]$ in $\mathscr{H}_i$ such that for some open neighborhood W in

$\mathrm{Aut}(\mathbb{P}^1)$ of the identity element we have

$$\mathscr{U} = \{[f_t \circ \varphi]: t \in U, \varphi \in W\}.$$

Clearly we may assume that $W \subset\subset \mathrm{Aut}(\mathbb{P}^1)$ and that the mapping κ: $U \times W \to \mathscr{U}$ defined by $(t, \varphi) \mapsto [f_t \circ \varphi]$ is a biholomorphic transformation (replacing U by some $U' \subset\subset U$ if necessary). Suppose σ: $S \mapsto \mathscr{H}_i/\mathrm{Aut}(\mathbb{P}^1)$ is not injective for any choice of S. Then there exists sequences $t_j \mapsto 0$ and $t_j' \mapsto 0$ such that $\sigma(f_{t_j}) = \sigma(f_{t_j'})$ and $t_j \neq t_j'$. In other words

$$f_{t_j'} = f_{t_j} \circ \varphi_j.$$

Since $t_j' \neq t_j$ and κ: $U \times W \to \mathscr{U}$ is a biholomorphism we know that $\varphi_j \in \mathrm{Aut}(\mathbb{P}^1) - W$. We assert

(∗) The set $\{\varphi_j\}$ is relatively compact in $\mathrm{Aut}(\mathbb{P}^1)$.

First, assuming (∗) we are going to derive the lemma. Passing to a subsequence we may assume that $\varphi_j \to \psi$ for some $\psi \in \mathrm{Aut}(\mathbb{P}^1) - W$. Then we have $f = f \circ \psi$ (where $f = f_0$), contradicting the fact that $\mathrm{Aut}(\mathbb{P}^1)$ acts freely on $\mathscr{H}_i$. Finally we prove (∗). Without loss of generality we may assume that $f(0)$ is a smooth point of $f(\mathbb{P}^1)$. Fix a coordinate open neighborhood $V_0 \subset\subset V$ of $f(0)$ in X such that $f(\Delta) \subset V$ and $f(\Delta) \cap V_0$ is a closed subvariety of V_0, i.e. $\partial f(\Delta) \cap V_0 = \varnothing$. Here Δ denotes a fixed disc in $\mathbb{C} \subset \mathbb{P}^1$ centered at the origin. Consider the set $E_0 \subset \mathrm{Aut}(\mathbb{P}^1)$ defined by

$$E_0 = \{\varphi \in \mathrm{Aut}(\mathbb{P}^1): (f \circ \varphi)(\Delta) \subset V \text{ and } \partial[(f \circ \varphi)(\Delta)] \cap V_0 = \varnothing\}.$$

Then clearly $\overline{E}_0$ is a compact subset of $\mathrm{Aut}(\mathbb{P}^1)$. Recall that f is injective and $f(0)$ is a smooth point of $f(\mathbb{P}^1)$. For $t \in U$ sufficiently close to 0, we still have $f_t(\Delta) \subset V$ and $\partial[f_t(\Delta)] \cap V_0 = \varnothing$. Defining E_t similarly we have $E_t \subset\subset \mathrm{Aut}(\mathbb{P}^1)$ and that moreover, shrinking U if necessary

$$\bigcup_{t \in U} E_t \subset\subset \mathrm{Aut}(\mathbb{P}^1).$$

Clearly, this implies (∗) and proves the lemma.

This proves that $\mathscr{Q}_i = \mathscr{H}_i/\mathrm{Aut}(\mathbb{P}^1)$ is a complex manifold for $1 \leqslant i \leqslant m$. Using this we can prove

Proposition (2.2.2). *Let* σ: $\hat{\mathscr{C}} \to \mathscr{C}$ *be the normalization of* $\mathscr{C}$. *Then* $\hat{\mathscr{C}}$ *is a connected nonsingular* $(q + n - 3)$*-dimensional projective variety. Moreover,* $m = 1$ *and the holomorphic map* $\hat{\Phi}$: $\mathscr{H} \mapsto \hat{\mathscr{C}}$ *obtained by passing to normalizations (of* Φ: $\mathscr{H} \to \mathscr{C}$*) is a submersion and realizes* $\mathscr{H}$ *as an* $\mathrm{Aut}(\mathbb{P}^1)$*-principal bundle over* $\hat{\mathscr{C}}$.

Proof. By the preceding discussion it suffices to show that $\mathscr{C}$ is irreducible and that $m = 1$. In this case ν_1: $\mathscr{Q}_1 \mapsto \mathscr{C}$ can be identified with the normalization σ: $\hat{\mathscr{C}} \mapsto \mathscr{C}$. Then the mapping $\hat{\Phi}$: $\mathscr{H} \to \hat{\mathscr{C}}$ is the natural map $\mathscr{H} = \mathscr{H}_1 \mapsto$

$\mathscr{H}_1/\mathrm{Aut}(\mathbb{P}^1) = \mathscr{Q}_1 \cong \hat{\mathscr{C}}$. Suppose $\mathscr{C}$ is irreducible. We are going to derive a contradiction. First, we assert that the lifting property (L) below from cycles to maps $f\colon \mathbb{P}^1 \mapsto X$ contradicts the reducibility of $\mathscr{C}$ (i.e., the disconnectedness, of $\hat{\mathscr{C}}$).

(L) Let $\Gamma \subset \mathscr{C}$ be a connected projective curve, and let $\sigma_\Gamma\colon \hat{\Gamma} \mapsto \Gamma$ be the normalization of Γ. Fix $b \in \Gamma$ and let $f\colon \mathbb{P}^1 \mapsto X$ be a rational curve such that $[f(\mathbb{P}^1)] = b \in \Gamma$. Suppose $\hat{\gamma}_0 \in \hat{\Gamma}$ is any point such that $\sigma_\Gamma(\hat{\gamma}_0)$ is distinct from b. Then there exists a holomorphic map $\theta\colon \hat{\Gamma} - \{\hat{\gamma}_0\} \mapsto \mathscr{H}$ such that $\Phi(\theta(\hat{\gamma})) = \sigma_\Gamma(\hat{\gamma})$ for every $\hat{\gamma} \in \hat{\Gamma} - \{\hat{\gamma}_0\}$ and such that $\theta(\hat{b}) = f$ for some $\hat{b} \in \hat{\Gamma}$ satisfying $\sigma_\Gamma(\hat{b}) = b$.

Assuming (L) for the time being, we are going to contradict the reducibility of $\hat{\mathscr{C}}$. If $\hat{\mathscr{C}}$ is reducible, there exists two points $\hat{b}_1$, $\hat{b}_2$ on connected components $\hat{\mathscr{C}}_1$, $\hat{\mathscr{C}}_2$ respectively such that $\hat{\mathscr{C}}_1 \neq \hat{\mathscr{C}}_2$ and such that both points correspond to the same cycle $[C] \in \mathscr{C}$. Let $f\colon \mathbb{P}^1 \mapsto X$ be a holomorphic rational curve such that $f(\mathbb{P}^1) = C$. By the lifting property (L) every cycle $[C_i]$, $\mathscr{C}_i$, $i = 1, 2$, can be lifted to the connected component $\mathscr{H}_1$ of $\mathscr{H}$ containing $[f]$. Thus $\Phi(\mathscr{H}_1) \supset \mathscr{C}_1 \cup \mathscr{C}_2$ but $\Phi(\mathscr{H}_1) \not\subset \mathscr{C}_i$, $i = 1, 2$, which is clearly a contradiction to the irreducibility of $\mathscr{H}_1$.

We assert that the irreducibility of $\mathscr{C}$ implies immediately $m = 1$ and thus Proposition (2.2.2). In fact, since $g\colon \mathbb{P}^1 \mapsto X$ is injective generically for every $[g] \in \mathscr{H}$ we see by passing to normalizations that $[g(\mathbb{P}^1)] = [g'(\mathbb{P}^1)]$ if and only if $g' = g \circ \varphi$ for some $\varphi \in \mathrm{Aut}(\mathbb{P}^1)$, so that $\Phi(\mathscr{H}_i) \cap \Phi(\mathscr{H}_j) = \varnothing$ for $1 \leqslant i < j \leqslant m$. On the other hand, the lifting property (L) implies readily that $\Phi(\mathscr{H}_i) = \mathscr{C}$ for $1 \leqslant i \leqslant m$. It follows immediately that $m = 1$. The preceding argument implies that the natural map $\nu_1\colon \mathscr{Q}_1 \mapsto \hat{\mathscr{C}}$ is injective. It is also surjective since $\bigcup_{1 \leqslant i \leqslant m} \nu_i(\Theta_i) = \mathscr{C}$ and $m = 1$. Thus, ν_1 is a bijective holomorphic map between normal complex spaces, hence necessarily a biholomorphism. The proof of Proposition (2.2.2) is finished by replacing $\mathscr{Q}_1$ by $\hat{\mathscr{C}}$.

We proceed finally to prove the lifting property (L) from cycles $[C] \in \mathscr{C}$ to maps $f\colon \mathbb{P}^1 \mapsto X$. Let $\mathscr{E}_\Gamma$ be the part of $\mathscr{E}$ sitting above Γ. Let $\hat{\mathscr{E}}$ be a normalization of $\mathscr{E}$ and $\hat{\pi}_\Gamma\colon \hat{\mathscr{E}}_\Gamma \mapsto \hat{\Gamma}$ be obtained from $\pi_\Gamma\colon \mathscr{E}_\Gamma \mapsto \Gamma$ by passing to normalizations. Let $\mu\colon \Sigma \mapsto \hat{\mathscr{E}}_\Gamma$ be a desingularization of $\hat{\mathscr{E}}_\Gamma$. By the theorem of Enriques-Noether (Theorem C) on ruled surfaces, Σ admits a minimal model Σ_0 such that the mapping $\hat{\pi}_\Gamma \circ \mu\colon \mathscr{S} \mapsto \hat{\Gamma}$ induces a ruling $\Sigma_0 \mapsto \hat{\Gamma}$. Thus, Σ_0 is obtained from Σ by successively blowing down exceptional curves of self-intersection (-1) that lie above a single point of $\hat{\Gamma}$. Since $\hat{\mathscr{E}}_\Gamma$ has only isolated singularities it is clear that there is a homeomorphism of $\hat{\mathscr{E}}_\Gamma$ onto Σ_0 which is a biholomorphism outside a finite number of points. Since $\hat{\mathscr{E}}_\Gamma$ and Σ_0 are both normal this implies by the Riemann Extension Theorem that $\hat{\mathscr{E}}_\Gamma$ and Σ_0 are biholomorphic, i.e. $\hat{\pi}_\Gamma\colon \hat{\mathscr{E}}_\Gamma \mapsto \hat{\Gamma}$ already realizes $\hat{\mathscr{E}}_\Gamma$ as a holomorphic $\mathbb{P}^1$-bundle over $\hat{\Gamma}$. For any $\hat{\gamma}_0 \in \hat{\Gamma}$, $\Gamma' = \hat{\Gamma} - \{\hat{\gamma}_0\}$ is an affine algebraic curve and the $\mathbb{P}^1$-bundle $\hat{\pi}_\Gamma^{-1}(\Gamma')$ over Γ' is analytically (even algebraically) trivial. Thus, we have a biholomorphism $\hat{\pi}_\Gamma^{-1}(\Gamma') \cong \Gamma' \times \mathbb{P}^1$ as $\mathbb{P}^1$-bundles. The natural mapping $\hat{\pi}_\Gamma^{-1}(\Gamma') \mapsto \hat{\mathscr{E}}_\Gamma \mapsto \mathscr{E}_\Gamma \subset \mathscr{C} \times X \mapsto X$ can be regarded as a family of holomorphic maps $g_{\gamma'}$: $\mathbb{P}^1 \mapsto X$ parametrized by $\gamma' \in \Gamma'$. Certainly, for any $\hat{b} \in \Gamma'$, $\hat{b} \neq \hat{\gamma}_0$ sitting above b we have $[g_{\hat{b}}(\mathbb{P}^1)] = [f(\mathbb{P}^1)]$. Fix one such $\hat{b}$ and consider $\tilde{g}_{\Gamma'}$: $\mathbb{P}^1 \mapsto X$ defined by $\tilde{g}_{\gamma'} = g_{\gamma'} \circ \varphi$ for $\varphi \in \mathrm{Aut}(\mathbb{P}^1)$ such that $g_{\hat{b}} \circ \varphi = f$. Then the family of maps $\{\tilde{g}_{\gamma'}: \gamma' \in \Gamma'\}$ yields the lifting property (L) as asserted.

(2.3) Construction of $\mathscr{S}$: Part I. As above let (X, h) be a compact Kähler manifold satisfying $b_2(X) = 1$ carrying nonnegative holomorphic bisectional curvature, positive Ricci curvature, and positive holomorphic sectional curvature. Let f_0: $\mathbb{P}^1 \mapsto X$ be a given rational curve of minimal degree and $\mathscr{H}$, $\hat{\mathscr{C}}$, π: $\hat{\mathscr{E}} \mapsto \hat{\mathscr{C}}$ be constructed as above using f_0. Suppose $f_0^*(K_X^{-1}) = \mathscr{O}(q)$ with $0 < q \leqslant n + 1$. Then it follows readily from the Semi-Continuity Theorem of Grauert (part of Theorem E) that there exists an $\mathrm{Aut}(\mathbb{P}^1)$-invariant subvariety $E \subset \mathscr{H}$ such that for all $[f] \in \mathscr{H} - E$, we have the splitting $f^*(T_X) \cong \mathscr{O}(a_1) \oplus \cdots \oplus \mathscr{O}(a_n)$, $a_1 \geqslant \cdots \geqslant a_n$, with a_p fixed, $1 \leqslant k \leqslant n$, independent of f. Since the natural map $T_{\mathbb{P}^1} \hookrightarrow f^*(T_X)$ induced by df: $T_{\mathbb{P}^1} \mapsto T_X$ is nontrivial and $T_{\mathbb{P}^1} \cong \mathscr{O}(2)$ it follows immediately that $a_1 \geqslant 2$. We are going to define $\mathscr{S} \subset \mathbb{P}T_X$ differently for the two cases (i) $2 \leqslant q < n + 1$ and (ii) $q = n + 1$, and $X \not\cong \mathbb{P}^n$. In the former case we have $a_n = 0$. Let $\mathscr{S}_0 \subset \mathbb{P}T_X$ be the collection of all tangent directions $[\alpha] \in \mathbb{P}T_x(X)$, $x \in X$, to rational cycles $[C] \in \mathscr{C}$ of minimal degree at regular points x of C. (Here we denote by $[\alpha] \in \mathbb{P}T_x(X)$ the canonical image of $\alpha \in T_x(X) - \{0\}$.) Let $\mathscr{S}$ be the closure of $\mathscr{S}_0$ in $\mathbb{P}T_X$. Because $f^*(T_X)$ has trivial components in the splitting as direct sums of holomorphic line bundles over $\mathbb{P}^1$ for any $[f] \in \mathscr{H}$, we can show that $\mathscr{S} \subset \mathbb{P}T_X$ is a proper subvariety. In the case $q = n + 1$ we know by Mori [20] that $X \cong \mathbb{P}^n$ as long as there exists some point $x \in X$ such that we have $f^*(T_X) \cong \mathscr{O}(2) \oplus \mathscr{O}(1)^{n-1}$ for all rational curves $[f] \in \mathscr{H}$ passing through x. In the event $X \not\cong \mathbb{P}^n$, we know that for every $x \in X$ there exists $[f] \in \mathscr{H}$ such that $f^*(T_X)$ has trivial components. If $f^*(T_X)$ has trivial components for all $[f] \in \mathscr{H}$ we construct $\mathscr{S}$ as previously. Otherwise $f^*(T_X) \cong \mathscr{O}(2) \oplus \mathscr{O}(1)^{n-1}$ generically. We call curves $[f] \in \mathscr{H}$, for which $f^*(T_X)$ has trivial components, special rational curves. Then, we define $\mathscr{S}$ just as in the case $2 \leqslant q < n + 1$ by restricting our construction to special rational cycles $[C] \in \mathscr{C}$. This way we are going to show that $\mathscr{S} \subset \mathbb{P}T_X$ is a subvariety of codimension 1. We remark that eventually it will follow from Theorem 1 that the assumption $q = n + 1$ implies automatically $X \cong \mathbb{P}^n$, so that the a priori possibility (ii) $q = n + 1$, $X \not\cong \mathbb{P}^n$ does not happen.

In the notations adopted above we are going to prove

Proposition (2.3). *Suppose* $2 \leqslant q \leqslant n+1$ *and* $f^*(T_X) = \mathcal{O}(a_1) \oplus \cdots \oplus \mathcal{O}(a_{n-l}) \oplus \mathcal{O}^l$ *for a generic* $[f] \in \mathcal{H}_1$ *with* $a_1 \geqslant \cdots \geqslant a_{n-l} \geqslant 1$ *and* $l \geqslant 1$. *Then* $\mathcal{S} \subset \mathbb{P}T_X$ *is a closed subvariety of* $\mathbb{P}T_X$ *of codimension* l. *Suppose* $q = n+1$, $X \not\cong \mathbb{P}^n$, *and* $f^*(T_X) = \mathcal{O}(2) \oplus \mathcal{O}(1)^{n-1}$ *for a generic* $[f] \in \mathcal{H}_1$. *Then, for a generic special rational curve* $[g] \in \mathcal{H}_1$ *of minimal degree,* $g^*(T_X) = \mathcal{O}(2)^2 \oplus \mathcal{O}(1)^{n-3} \oplus \mathcal{O}$. *Moreover,* $\mathcal{S} \subset \mathbb{P}T_X$ *is a (possibly singular) hypersurface in* $\mathbb{P}T_X$.

Proof: *Part* I. In this section we will only prove the first part of Proposition (2.3), i.e., we assume that for any $[f] \in \mathcal{H}$, $f^*(T_X)$ has trivial components. We assert first of all that in general the normalization $\hat{\pi}: \hat{\mathcal{E}} \mapsto \hat{\mathcal{C}}$ of $\pi: \mathcal{E} \mapsto \mathcal{C}$ realizes $\hat{\mathcal{E}}$ as a holomorphic $\mathbb{P}^1$-bundle over the compact complex manifold $\hat{\mathcal{C}} = \mathcal{Q} = \mathcal{H}/\mathrm{Aut}(\mathbb{P}^1)$, $\mathcal{H} = \mathcal{H}_1$. To see this consider the free right action of $\mathrm{Aut}(\mathbb{P}^1)$ on $\mathcal{H} \times \mathbb{P}^1$ given by $\varphi([f]; z)) = ([f \circ \varphi], \phi^{-1}(z))$. Then $\mathcal{P} = \mathcal{H} \times \mathbb{P}^1/\mathrm{Aut}(\mathbb{P}^1)$ can be given the structure of a $(q+n-1)$-dimensional compact complex manifold. In fact, $\mathcal{P} = \mathcal{H} \times_{\mathrm{Aut}(\mathbb{P}^1)} \mathbb{P}^1$ is the $\mathbb{P}^1$-bundle induced by the principal $\mathrm{Aut}(\mathbb{P}^1)$-bundle $\mathcal{H}$. On the other hand the mapping $\mathcal{H} \times \mathbb{P}^1 \mapsto \mathcal{E}$ defined by the assignment $([f], z) \mapsto ([f(\mathbb{P}^1)], f(z)) \in \mathcal{E} \subset \mathcal{C} \times X$ is clearly invariant under the $\mathrm{Aut}(\mathbb{P}^1)$-action on $\mathcal{H} \times \mathbb{P}^1$. This yields a surjective holomorphic map $\mathcal{P} \mapsto \mathcal{E}$ with finite fibers which is generically injective. Then, passing to normalizations we see that $\hat{\mathcal{E}}$ is biholomorphic to the complex manifold $\mathcal{P}$ and that $\hat{\pi}: \hat{\mathcal{E}} \mapsto \hat{\mathcal{C}}$ realizes $\hat{\mathcal{E}}$ as a holomorphic $\mathbb{P}^1$-bundle.

Let now $\mathcal{L} \mapsto \hat{\mathcal{E}}$ be the holomorphic line bundle $\hat{\mathcal{E}}$ whose restriction to every $\mathbb{P}^1$-fiber of $\hat{\pi}: \hat{\mathcal{E}} \mapsto \hat{\mathcal{C}}$ is the tangent bundle along the fiber. Let $\mathcal{V} \subset \hat{\mathcal{E}}$ be the subvariety consisting of all $v \in \hat{\mathcal{E}}$ such that $d\theta|_{\mathcal{L}_v} \equiv 0$. The mapping $d\theta|_{\mathcal{L}}$: $\mathcal{L} \mapsto T_X$ induces a map $\Theta: \mathbb{P}(\mathcal{L}|_{\hat{\mathcal{E}}-\mathcal{V}}) \mapsto \mathbb{P}T_X$. Clearly $\mathcal{S}$ is nothing but the closure of $\Theta(\mathbb{P}(\mathcal{L}|_{\hat{\mathcal{E}}-\mathcal{V}})$ in $\mathbb{P}T_X$. To see that $\mathcal{S}$ is a closed subvariety it suffices to observe by using local coordinates that Θ can be regarded as a meromorphic map on $\mathbb{P}(\mathcal{L})$. Thus $\overline{\mathrm{Graph}(\Theta)} \subset \mathbb{P}(\mathcal{L}) \times \mathbb{P}T_X$ is a subvariety and $\mathcal{S}$ is the projection of $\overline{\mathrm{Graph}(\Theta)}$ into $\mathbb{P}T_X$.

We are going to determine the codimension of $\mathcal{S}$ in $\mathbb{P}T_X$. Let $\mathcal{F} \mapsto \hat{\mathcal{E}}$ be the holomorphic vector bundle on $\hat{\mathcal{E}}$ such that for each $\mathbb{P}^1$-fiber $\hat{\pi}^{-1}(b)$ of $\hat{\mathcal{E}}$ representing $[C] \in \mathcal{C}$, $\mathcal{F}|_{\hat{\pi}^{-1}(b)} \cong f^*T_X$ for any $f: \mathbb{P}^1 \mapsto X$ realizing $[C]$. It is obvious how the f^*T_X can be pieced together to form a holomorphic vector bundle. By Theorem C on a Zariski open subset Z of $\hat{\mathcal{C}}$ we have $\mathcal{F}|_{\hat{\pi}^{-1}(b)} \cong \mathcal{O}(a_1) \oplus \cdots \oplus \mathcal{O}(a_{n-l}) \oplus \mathcal{O}^l$ with $a_1 \geqslant \cdots \geqslant a_{n-l} > 0$ fixed independent of $b \in \hat{\mathcal{C}}$. We write $\mathcal{S} = \bigcup_{x \in X} \mathcal{S}_x$ with $\mathcal{S}_x = \mathcal{S} \cap \mathbb{P}T_x(X)$. Consider $\mathcal{C}_0(x)$, the subvariety of $\mathrm{Chow}(K_X^{-1}; q)$ consisting of cycles passing through x, and write $\mathcal{C}(x) = \mathcal{C}_0(x) \cap \mathcal{C}$. Consider now an arbitrary point x on X and an arbitrary cycle $[C_0] \in \mathcal{C}(x)$ such that $[C_0]$ corresponds to an element of Z. Assume also that x is a smooth point of C_0 and that $\alpha_0 = [T_x(C_0)] \in \mathscr{S}$ is a smooth point of $\mathscr{S}$ such that the projection map $\chi\colon \mathscr{S} \mapsto X$ is a submersion at x. We are going to show that $\mathscr{S}_x$ is of codimension l in $\mathbb{P}T_x(X)$ at α_0.

Let $\mathscr{H}(x) \subset \mathscr{H}$ consist of all $[f] \in \mathscr{H}$ such that $f(0) = x$. Write $\mathscr{I}_0$ for the ideal sheaf of $\{0\}$ in $\mathbb{P}^1$. Then, by deformation theory (Theorem B, relative version) $[f]$ is a smooth point of $\mathscr{H}(x)$ as soon as $H^1(\mathbb{P}^1, f^*T_X \otimes \mathscr{I}_0) = 0$. In this case one can identify the tangent space of $\mathscr{H}(x)$ at $[f]$ by $T_{[f]}\mathscr{H}(x) \cong H^0(\mathbb{P}^1, f^*T_X \otimes \mathscr{I}_0)$. Clearly $\mathscr{I}_0 \cong \mathscr{O}(-1)$ on $\mathbb{P}^1$. Since $f^*(T_X) \cong \mathscr{O}(a_1) \oplus \cdots \oplus \mathscr{O}(a_{n-l}) \oplus \mathscr{O}^l$ with all $a_k > 0$, and since $H^1(\mathbb{P}^1, \mathscr{O}(-1)) = 0$ and $H^0(\mathbb{P}^1, \mathscr{O}(\gamma)) = \gamma + 1$ for $\gamma > 0$, we obtain that $\mathscr{H}(x)$ is smooth of dimension $\sum_{k=1}^{n-l} a_k = q < n + 1$. Consider now the variation of $\alpha = [T_x(C)]$ as $[C]$ varies in a neighborhood of $[C_0]$ in $\mathscr{C}(x)$. Suppose $C_0 = f_0(\mathbb{P}^1)$. It is equivalent to consider the variation of $df(\frac{\partial}{\partial z})$ for a nonzero holomorphic vector $\frac{\partial}{\partial z}$ at $0 \in \mathbb{P}^1$, as f varies in a neighborhood of f_0. Given any section s of $H^0(\mathbb{P}^1, f^*T_X \otimes \mathscr{I}_0) \cong T_{[f]}\mathscr{H}(x)$ there is a one-parameter family $\{f_t\}$ of maps in $\mathscr{H}(x)$ such that f_0 is as given and $\frac{\partial}{\partial t} f_t(z)|_{t=0} = s(z)$. Write $\alpha_0 = df_0(\frac{\partial}{\partial z})$. The tangent to $[f_t(\mathbb{P}^1)]$ at x is given by

$$T_x\big(f_t(\mathbb{P}^1)\big) = \mathbb{C}\big(\alpha_0 + tf_*(s(0)) + O(t^2)\big).$$

We can identify $T_{\alpha_0}(\mathbb{P}T_x(X))$ with $\mathbb{C}$-linear subspaces of $T_x(X)$ containing $\mathbb{C}\alpha_0$ and thus with $T_x(X)/\mathbb{C}\alpha_0$. With this identification it is then clear that

$$T_{\alpha_0}(\mathscr{S}_x) = \Big\{\mathbb{C}\alpha_0 + \sum_{s \in \Gamma} \mathbb{C} f_*(s(0))\Big\}/\mathbb{C}\alpha_0.$$

Recall that $f^*T_X \cong \mathscr{O}(a_1) \oplus \cdots \oplus \mathscr{O}(a_{n-l}) \oplus \mathscr{O}^l$ with $a_1 \geqslant \cdots \geqslant a_{n-l} > 0$. Let v_k be a basis of the complex line at 0 corresponding to the kth component line bundle $\mathscr{O}(a_k)$. (We write $a_{n-l+1} = \cdots = a_n = 0$.) Then observing that

$$\alpha_0 = df_0\left(\frac{\partial}{\partial z}\right) \in \sum_{k=1}^{n-l} \mathbb{C} f_*(v_k) \subset T_x(X)\Big),$$

we obtain

$$\begin{aligned} T_{\alpha_0}(\mathscr{S}_x) &= \Big\{\mathbb{C}\alpha_0 + \sum_{k=1}^{n-l} \mathbb{C} f_*(v_k)\Big\}/\mathbb{C}\alpha_0 \\ &= \sum_{k=1}^{n-l} \mathbb{C} f_*(v_k)/\mathbb{C}\alpha_0. \end{aligned}$$

In fact there exists a section $s \in H^0(\mathbb{P}^1, f^*T_X \otimes \mathscr{I}_0)$ with $s(0) = v_k$ if and only if $a_k > 0$. Thus $T_{\alpha_0}(\mathscr{S}_x)$ is of codimension l in $T_{\alpha_0}(\mathbb{P}T_x(X))$ so that $\mathscr{S}$ is of codimension l in $\mathbb{P}T_x$. The assertions of the first part of the proposition are thus completed.

(2.4) Construction of $\mathscr{S}$: Part II. In this section we are going to prove Proposition (2.3) for the case $q = n + 1$, $X \not\cong \mathbb{P}^n$, and $f^*(T_X) \cong \mathcal{O}(2) \oplus \mathcal{O}(1)^{n-1}$ for a generic $[f] \in \mathscr{H}$. As in (2.3) consider the holomorphic $\mathbb{P}^1$-bundle $\hat{\pi}$: $\hat{\mathscr{E}} \mapsto \hat{\mathscr{C}}$ given by $\hat{\mathscr{E}} \cong \mathscr{H} \times_{\mathrm{Aut}(\mathbb{P}^1)} \mathbb{P}^1$. Let $\Sigma \subset \hat{\mathscr{C}}$ correspond to the set of special rational curves. Consider the meromorphic mapping Θ: $\hat{\mathscr{E}} \mapsto \mathbb{P}T$ as in (2.3). Our main question is to study the map $\Theta|_{\hat{\mathscr{E}}_\Sigma}$, where $\hat{\mathscr{E}}_\Sigma = \hat{\pi}^{-1}(\Sigma)$. We are going to prove that Σ is of codimension one in $\hat{\mathscr{C}}$ and that $\Theta|_{\hat{\mathscr{E}}_\Sigma}$ is of maximal rank $= 2n - 2$. By Mori [20] we may assume without loss of generality that for each $x \in X$ there exists some rational curve $[f] \in \mathscr{H}(x)$ such that $f^*(T_X) \not\cong \mathcal{O}(2) \oplus \mathcal{O}(1)^{n-1}$. Let $\mathscr{V}_\Sigma \subset \hat{\mathscr{E}}_\Sigma$ consist of points where Θ is not holomorphic. Given $\Theta|_{\hat{\mathscr{E}}_\Sigma}$ is of rank $2n - 2$ we conclude then that for $\mathscr{S} = \overline{\Theta(\hat{\mathscr{E}}_\Sigma - \mathscr{V}_\Sigma)}$, $\mathscr{S}_x = \mathscr{S} \cap \mathbb{P}T_x$ is generically of codimension one in $\mathbb{P}T_X$, as $\mathscr{S}$ projects to all of X.

We proceed to prove

Lemma (2.4.1). *$\Sigma \subset \hat{\mathscr{E}}$ is of codimension one.*

Proof. Equivalently we need to prove $\hat{\mathscr{E}}_\Sigma \subset \hat{\mathscr{C}}$ is of codimension one. The set $\mathscr{V} \subset \hat{\mathscr{E}}$ where Θ is not holomorphic consists precisely of points corresponding to singular points of rational cycles $[f(\mathbb{P}^1)] \in \mathscr{C}$, excluding self-intersections of smooth pieces. We are going to show that, on $\hat{\mathscr{E}} - \mathscr{V}$, $\hat{\mathscr{E}}_\Sigma$ can be identified with the branching locus of Θ. Since $f^*T_X \cong \mathcal{O}(2) \oplus \mathcal{O}(1)^{n-1}$ for a generic rational curve $[f] \in \mathscr{H}$, the argument of (2.3) shows that Θ is of rank $2n - 1$, $\overline{\Theta(\hat{\mathscr{E}} - \mathscr{V})} = \mathbb{P}T_X$. But $\dim_{\mathbb{C}} \hat{\mathscr{E}} = 1 + \dim_{\mathbb{C}} \hat{\mathscr{C}} = 1 + (n + 1) + n - 3 = 2n - 1$, Θ is a local biholomorphism at generic points. Thus $\hat{\mathscr{E}}_\Sigma - \mathscr{V}$ is of codimension one in $\hat{\mathscr{E}}$. Since clearly $\hat{\mathscr{E}}_\Sigma = \overline{\hat{\mathscr{E}}_\Sigma - \mathscr{V}}$ (every rational cycle contains at least one smooth point!), we can thus conclude the proof of the lemma. We identify $\hat{\mathscr{E}}$ with $\mathscr{P} = \mathscr{H} \times_{\mathrm{Aut}(\mathbb{P}^1)} \mathbb{P}^1 = \mathscr{H} \times \mathbb{P}^1/R$ for the equivalence relation $([f], z) \overset{R}{\sim} ([g], w)$ if and only if $g = f \circ \varphi$ and $w = \varphi^{-1}(z)$ for some $\varphi \in \mathrm{Aut}(\mathbb{P}^1)$. Then, at $\{[f], z\} \overset{\mathrm{def}}{=} ([f], z) \bmod R$ we can identify $T_{\{[f], z\}}(\hat{\mathscr{E}})$ with a quotient space of $T_{[f]}(\mathscr{H}) + T_z(\mathbb{P}^1)$. More precisely, letting τ_f: $H^0(\mathbb{P}^1, T_{\mathbb{P}^1}) \mapsto H^0(\mathbb{P}^1, f^*T_X) \cong T_{[f]}(\mathscr{H})$ be induced by f and $\tau_f'(s) = (\tau_f(s), -s(z))$, then

$$T_{\{[f], z\}}(\hat{\mathscr{E}}) \cong H^0(\mathbb{P}^1, f^*T_X) + T_z(P^1)/\tau_f'(H^0(\mathbb{P}, T_{\mathbb{P}^1})).$$

We are going to compute $\mathrm{Ker}\, d\Theta(\{[f], z\})$. On some open neighborhood U of $x = f(z)$ identify $\mathbb{P}T_U$ with $U \times \mathbb{P}^{n-1}$, in terms of local holomorphic coordinates. Then, we can write $\Theta = (\sigma, \Theta')$ where $\sigma(\{[g], w\}) = g(w)$ is the composition of Θ with the natural projection $\mathbb{P}T_U \mapsto \mathbb{P}^{n-1}$. For $(s, v) \in H^0(\mathbb{P}^1, f^*T_X) + T_z(\mathbb{P}^1)$ write $[s, v]$ for $(s, v) \bmod \tau_f'(H^0(\mathbb{P}^1, T_{\mathbb{P}^1})$. Clearly

$[s, v] = [s', 0]$ for some $s' \in H^0(\mathbb{P}^1, f^*T_X)$. We compute first $\operatorname{Ker} d\sigma(\{[f], z\})$. It is easy to verify that we have the formula

$$d\sigma([s, v]) = s(z) + f_*(v),$$

so that $d\sigma([s, 0]) = 0$ if and only if $s(z) = 0$. Thus, we have

$$\begin{aligned}\operatorname{Ker} d\Theta\{[f], z\} &\subset \{[s, 0]: s \in H^0(\mathbb{P}^1, f^*T_X) \text{ vanishes at } z\} \\ &\cong H^0(\mathbb{P}^1, f^*T_X \otimes \mathscr{I}_z)/\tau_f(H^0(\mathbb{P}^1, T_{\mathbb{P}^1} \otimes \mathscr{I}_z)) \\ &\cong T_{[f']}(\mathscr{H}(x)),\end{aligned}$$

where $\mathscr{I}_z$ = ideal sheaf of $\{z\}$ on $\mathbb{P}^1$ and $[f'] \in \mathscr{H}(x)$ is the element of $\mathscr{H}(x)$ defined by f, sending the distinguished point z to x. (In (2.3) we used 0 to denote the distinguished point.) Now, $d\Theta|_{\{[f], z\}}([s, 0]) = 0$ if and only if $s(z) = 0$ and $d\Theta'([s, 0]) = 0$. For any $s \in H^0(\mathbb{P}^1, f^*T_X \otimes \mathscr{I}_z)$, there exists a complex one-parameter family $\{f_t: t \in \Delta(\varepsilon)\}$, $\Delta(\varepsilon)$ denoting the disc of radius ε centered at 0, such that $f_0 = f$, $f_t(z) = x$, and $\partial f_t/\partial t = s$ on $\mathbb{P}^1$. (Recall that $H^1(\mathbb{P}^1, f^*T_X \otimes \mathscr{I}_z) = 0$.) Let w be a local parameter of $\mathbb{P}^1$ at z with $w(z) = 0$ and suppose $(\partial f/\partial w)(z) = \eta \neq 0$. We have assumed that f is unramified at z. Choose local holomorphic coordinates $(\zeta_1, \cdots, \zeta_n)$ at x such that $\zeta_i(x) = 0$, $1 \leqslant i \leqslant n$, and $\eta = c\partial/\partial\zeta_1$, $c \neq 0$. Then we can write

$$f_t(w) = f(w) + t\chi(w) + O(|t|^2),$$

$$f_*s(w) = \sum \chi_i \frac{\partial}{\partial \zeta_i}, \qquad \chi = (\chi_1, \cdots, \chi_n),$$

$$\frac{\partial f_t}{\partial w}(z) = c\frac{\partial}{\partial \zeta_1} + \frac{\partial \chi}{\partial w}(z) + O(|t|^2).$$

Thus

$$\Theta'(\{[f_t, z]\}) = \left[c\frac{\partial}{\partial \zeta_1} + t\frac{\partial \chi}{\partial w}(z) + O(|t|^2)\right] \in \mathbb{P}^{n-1},$$

giving

$$d\Theta'|_{\{[f_t], z\}}([s, 0]) = \frac{1}{c}\frac{\partial \chi}{\partial w}(z) \bmod \mathbb{C}\frac{\partial}{\partial \zeta_1} \in T_{[\partial/\partial\zeta_1]}(\mathbb{P}^{n-1}).$$

In particular for $s \in H^0(\mathbb{P}^1, f^*T_X \otimes \mathscr{I}_z)$, $[s, 0] \in \operatorname{Ker} d\Theta'|_{\{[f_t], z\}}$ if and only if $(\partial\chi/\partial w)(z) = 0$. This happens if and only if s vanishes to an order $\geqslant 2$ at z. Hence, $\operatorname{Ker} d\Theta_{\{[f_t], z\}} \neq 0$ if and only if

$$H^0(\mathbb{P}^1, f^*T_X \otimes \mathscr{I}_z^2)/\tau_f(H^0(\mathbb{P}^1, T_{\mathbb{P}^1} \otimes \mathscr{I}_z^2)) \neq 0.$$

For a generic $[f] \in \mathscr{H}$, $f^*T_X \cong \mathscr{O}(2) \oplus \mathscr{O}(1)^{n-1}$, so that all holomorphic sections s of f^*T_X vanishing to an order $\geqslant 2$ must come from $H^0(\mathbb{P}^1, T_{\mathbb{P}^1})$, while for a special rational curve $[f] \in \Sigma$ (and z a point where f is unramified), $H^0(\mathbb{P}^1, f^*T_X \otimes \mathscr{I}_z^2)$ has at least two linearly independent holomorphic sections, so that we have

$$H^0\big(\mathbb{P}^1, f^*T_X \otimes \mathscr{I}_z^2\big)/\tau_f\big(H^0\big(\mathbb{P}^1, T_{\mathbb{P}^1} \otimes \mathscr{I}_z^2\big)\big) \neq 0,$$

if and only if $[f] \in \Sigma$. This proves the assertion that on $\hat{\mathscr{E}} - \mathscr{V}$, $\hat{\mathscr{E}}_{\mathscr{V}}$ identifies with the branching locus of Θ. The proof of the lemma is complete.

The main point of the second part of Proposition (2.3) is the assertion that $\mathscr{S}$ is of (pure) codimension 1 in $\mathbb{P}T_X$. Given Lemma (2.4.1), we have to show that $\Theta|_{\hat{\mathscr{E}}_\Sigma - \mathscr{V}}$ is of maximal rank equal to $2n - 2$. We offer here a plausible argument and explain why it is insufficient. Suppose $\Theta|_{\hat{\mathscr{E}}_\Sigma - \mathscr{V}}$ is not of maximal rank. Then for a generic $[\alpha] \in \mathscr{S}$, there will be a positive-dimensional subvariety of $\hat{\mathscr{E}}_\Sigma - \mathscr{V}$ being mapped to $[\alpha]$ by Θ. Since X is projective algebraic, so is $\hat{\mathscr{E}}_\Sigma$. It is then easy to find a one-parameter family of special rational cycles $[C]$ and a corresponding $\mathbb{P}^1$-bundle E over a nonsingular algebraic curve Γ such that some distinguished holomorphic section Γ_0 of E is mapped to $[\alpha] \in \mathscr{S}_x$ under Θ. Suppose each C is nonsingular except for self-intersections. Consider the natural map σ: $E \mapsto X$. Then the normal bundle of Γ in E is holomorphically trivial because $d\sigma(N_{\Gamma|E}) = \mathbb{C}\alpha$ while Γ is blown down by the map σ to a point. This would contradict the criterion of Mumford-Grauert (Theorem D) on blowing down curves. It is possible to prove a priori that almost all special rational curves are unramified. However, one cannot conclude the existence of E because the mapping Θ is only meromorphic on $\hat{\mathscr{E}}_\Sigma$. It may well happen that every family Γ of such rational cycles thus obtained carries a rational curve singular at x even though most special rational curves are unramified. (Here and from now on a singular rational curve $[f]$ means an f: $\mathbb{P}^1 \mapsto X$ which is degenerate somewhere.) To overcome the problem arising from the indeterminacies of Θ, we are going to construct instead a one-parameter family of special rational cycles $[C]$ over a nonsingular algebraic curve Γ such that the corresponding $\mathbb{P}^1$-bundle E admits two distinguished holomorphic sections Γ_0 and Γ_∞ blown down by the natural map σ': $E \mapsto X$. As was shown in Mori [20], this would also contradict the criterion of Mumford-Grauert. In the simple form that we need, it can be seen simply from

Lemma (2.4.2). *Let π': $E \mapsto \Gamma$ be a holomorphic $\mathbb{P}^1$-bundle over a nonsingular algebraic curve Γ. Suppose there exist two nonintersecting holomorphic sections Γ_0 and Γ_∞ of E over Γ. Then*

$$\pi'_*\big(N_{\Gamma_0|E}\big) \cong \pi'_*\big(N_{\Gamma_\infty|E}\big)^{-1}.$$

Consequently $N_{\Gamma_0|E}$ and $N_{\Gamma_\infty|E}$ cannot be simultaneously negative.

Proof. $\Pi'|_{E-\Gamma_i}: E - \Gamma_i \mapsto \Gamma$ $(i = 0, \infty)$ can be regarded as a holomorphic line bundle with Γ_∞, Γ_0 resp. representing the zero section. The lemma then follows immediately by observing that the system of transition functions are inverses of each other.

In order to construct $\pi: E \mapsto \Gamma$ we first prove a criterion for its existence. Then we will deduce this criterion from the hypothesis that $\Theta|_{\hat{\mathscr{E}}_\Sigma - \mathscr{V}}$ is not of maximal rank.

Lemma (2.4.3). *Suppose for a generic special rational curve $f: \mathbb{P}^1 \mapsto X$ and for any two points $z_1, z_2 \in \mathbb{P}^1$ there exists $s \in H^0(\mathbb{P}^1, f^*T_X)$ verifying $s(z_1) = s(z_2) = 0$, $s \notin \tau_f(H^0(\mathbb{P}^1, T_{\mathbb{P}^1}))$, such that, after identifying $T_{f(\mathbb{P}^1)}(\hat{\mathscr{C}})$ with $H^0(\mathbb{P}^1, f^*T_X)/\tau_f(H^0(\mathbb{P}^1, T_{\mathbb{P}^1}))$, $s \bmod \tau_f(H^0(\mathbb{P}^1, T_{\mathbb{P}^1}))$ belongs to $T_{[f(\mathbb{P}^1)]}(\Sigma)$. Then there exists a one-parameter family of special rational cycles $[C]$ parametrized by a nonsingular algebraic curve Γ such that for the corresponding holomorphic $\mathbb{P}^1$-bundle $\pi: E \mapsto \Gamma$, there exist two nonintersecting holomorphic sections Γ_0 and Γ_∞ of E over Γ with negative normal bundles, contradicting with Lemma* (2.4.2).

Remark. In terms of the $\mathbb{P}^1$-bundle $\hat{\pi}: \hat{\mathscr{E}} \mapsto \hat{\mathscr{C}}$, $\pi': E \mapsto \Gamma$ is obtained by taking some possibly singular $\Gamma' \subset \hat{\mathscr{C}}$, and normalizing the $\mathbb{P}^1$-bundle $\hat{\pi}|_{\hat{\mathscr{E}}_{\Gamma'}}$: $\hat{\mathscr{E}}_{\Gamma'} \mapsto \Gamma'$, $\hat{\mathscr{E}} = \hat{\pi}^{-1}(\Gamma')$.

Proof of Lemma (2.4.3). We construct first of all some local one-parameter family of special rational cycles containing two distinct points $x, y \in X$. Consider the complex manifold $\mathscr{W} \subset \mathscr{H} \times X^2$ defined by

$$\mathscr{W} = \{([f], f(0), f(\infty)) : [f] \text{ is a special rational curve}\}.$$

At $w \in \mathscr{W}$, let $\delta(w)$ be the dimension of

$$T_{[f(\mathbb{P}^1)]}(\Sigma) \cap H^0(\mathbb{P}^1, f^*T_X \otimes \mathscr{I}_0 \otimes \mathscr{I}_\infty) \bmod \tau_g\big(H^0(\mathbb{P}^1, T_{\mathbb{P}^1} \otimes \mathscr{I}_0 \otimes \mathscr{I}_\infty)\big).$$

By the hypothesis, $\delta(w) \geqslant 1$ for a generic w. There is an open subset (in the complex topology) Ω of $\mathscr{W}$ such that Ω is nonsingular and $\delta(w)$ is a constant. Write $w = ([f], f(0), f(\infty))$. Shrinking Ω if necessary, one can define a holomorphic mapping $\mu: \Omega \mapsto T(\Sigma)$ such that

$$\mu(w) \in T_{[f(\mathbb{P}^1)]}(\Sigma) \cap H^0(\mathbb{P}^1, f^*T_X \otimes \mathscr{I}_0 \otimes \mathscr{I}_\infty) \bmod \tau_f\big(H^0(\mathbb{P}^1, \otimes \mathscr{I}_0 \otimes \mathscr{I}_\infty)\big)$$

and $\mu(w) \neq 0$ for all $w \in \Omega$. Consider $\mathscr{W}$ as a graph over Σ. For $w \in \Omega$ there is a bijection between $T_{[f(\mathbb{P}^1)]}(\Sigma)$ and $T_w(\mathscr{W})$. In this bijection $(s \bmod \tau_f(H^0(\mathbb{P}^1, T_{\mathbb{P}^1})) \in T_{[f(\mathbb{P}^1)]}(\Sigma)$ corresponds to

$$\big(s \bmod \tau_f\big(H^0(\mathbb{P}^1, T_{\mathbb{P}^1})\big), s(0), s(\infty)\big) \in T_w(\mathscr{W}) \subset T_w(\mathscr{H} \times X^2).$$

Since $\mu(w)$ is represented by some s vanishing at 0, it follows that $(\mu(w), 0, 0) \in T_w(\mathscr{W})$. The assignment $w \mapsto (\mu(w), 0, 0)$ can then be regarded as a nonvanishing holomorphic vector field on Ω. Fix some f_0 such that $w_0 = ([f_0], f_0(0), f_0(\infty))$ belongs to Ω. Choose furthermore f_0 so that $f_0^{-1}(f_0(0)) = \{0\}$ and $f_0^{-1}(f_0(\infty)) = \infty$. Let $\{w_t: |t| < \varepsilon\}$ be a holomorphic integral curve of the holomorphic vector field $w \mapsto (\mu(w), 0, 0)$. In other words,

$$\frac{\partial w_t}{\partial t} = (\mu(w_t), 0, 0), \qquad w_0 = ([f_0], f_0(0), f_0(\infty)).$$

In particular, writing $w_t = (f_t, f_t(0), f_t(\infty))$ we have

$$f_t(0) = f_0(0), \quad f_t(\infty) = f_0(\infty) \quad \text{for all } |t| < \varepsilon.$$

Thus, we have constructed a local family of special rational curves containing the distinct points $f_0(0)$ and $f_0(\infty)$. Consider now the subset $F \subset \Sigma$ consisting of all special rational cycles passing through $x = f_0(0)$ and $y = f_0(\infty)$. Our construction implies that the analytic subvariety $F \subset \Sigma$ is of positive dimension. Clearly, there exists a complete algebraic curve $\Gamma' \subset \Sigma$, possibly singular, such that $[f(\mathbb{P}^1)] \in \Gamma'$. (Recall that $\hat{\mathscr{C}}$ is algebraic since $\mathscr{C}$ is.) Let ν: $\Gamma \mapsto \Gamma'$ be the normalization of Γ'. Γ is nonsingular. Write $\hat{\mathscr{E}}_{\Gamma'} = \pi^{-1}(\Gamma')$. Then $\hat{\pi}|_{\hat{\mathscr{E}}_{\Gamma'}}$: $\mathscr{E}_{\Gamma'} \mapsto \Gamma'$ is a holomorphic $\mathbb{P}^1$-bundle. Write $E = \nu^*(\mathscr{E}_{\Gamma'})$. Then π: $E \mapsto \Gamma$ thus defined is a holomorphic $\mathbb{P}^1$-bundle over a complete nonsingular algebraic curve. Furthermore since $[f_0(\mathbb{P}^1)] \in \Gamma'$, $f_0^{-1}(x) = \{0\}$, and $f_0^{-1}(y) = \{\infty\}$ over a generic point $\gamma \in \Gamma$ there must exist unique points $a_\gamma, b_\gamma \in \pi^{-1}(\gamma)$ such that $f_\gamma(a_\gamma) = x$, $f_\gamma(b_\gamma) = y$ for the natural mapping σ': $E \mapsto X$ induced by the mapping σ: $\hat{\mathscr{E}} \mapsto X$. Thus for $(\sigma')^{-1}\{x\} = \Gamma_0$ and $(\sigma')^{-1}\{y\} = \Gamma_\infty$, $\pi'|_{\Gamma_0}$: $\Gamma_0 \to \Gamma$ and $\pi'|_{\Gamma_\infty}$: $\Gamma_\infty \to \Gamma$ must be one-to-one and hence a biholomorphism (by the Riemann extension theorem for bounded holomorphic functions, applied to inverse functions). Thus, Γ_0 and Γ_∞ are nonsingular. They must be nonintersecting by definition. The proof of Lemma (2.4.3) is complete.

Now we are going to verify the hypothesis of Lemma (2.4.3), assuming that $\Theta|_{\hat{\mathscr{E}}_\Sigma - \mathscr{V}}$ is not of maximal rank, i.e. $2n-2$, hence proving Proposition (2.3) by contradiction.

Proof of Proposition (2.3). We use the notations of Lemma (2.4.1). Recall that one can identify $T_{[f(\mathbb{P}^1)]}(\mathscr{C})$ with

$$H^0(\mathbb{P}^1, f^*T_X) + T_Z(\mathbb{P}^1)/\tau_f'\big(H^0(\mathbb{P}^1, T_{\mathbb{P}^1})\big).$$

With this identification we know from Lemma (2.4.1) that

$$\begin{aligned} \operatorname{Ker} d\Theta|_{\{[f], z\}} &\cong H^0(\mathbb{P}^1, f^*T_X \otimes \mathscr{I}_z^2)/\tau_f\big(H^0(\mathbb{P}^1, T_{\mathbb{P}^1} \otimes \mathscr{I}_z^2)\big) \\ &\hookrightarrow H^0(\mathbb{P}^1, f^*T_X) + T_Z(\mathbb{P}^1)/\tau_f'\big(H^0(\mathbb{P}^1, T_{\mathbb{P}^1})\big). \end{aligned}$$

Suppose now that $\Theta|_{\hat{\mathscr{E}}_\Sigma - \mathscr{V}}$ is not of maximal rank. This would mean that at each point $\{[f], z\} \in \hat{\mathscr{E}}_\Sigma - \mathscr{V}$, $\operatorname{Ker} d\Theta|_{\{[f],z\}} \cap T_{\{[f],z\}}(\hat{\mathscr{E}}_\Sigma) \neq \{0\}$. Let $[f(\mathbb{P}^1)] \in \Sigma$ be a smooth point. For every point $z \in \mathbb{P}^1$ where f is unramified, let $[s_z, 0] \in H^0(\mathbb{P}^1, f^*T_X \otimes \mathscr{I}_z^2) + \{0\})$ mod $\tau_f'(H^0(\mathbb{P}^1, T_{\mathbb{P}^1}))$ belong to $\ker d\Theta|_{\{[f],z\}} \cap T_{\{[f],z\}}(\hat{\mathscr{E}}_\Sigma)$. Let $\hat{\pi}_*\colon T_{\{[f],z\}}(\hat{\mathscr{E}}_\Sigma) \mapsto T_{[f(\mathbb{P}^1)]}(\Sigma)$ be induced by $\hat{\pi}\colon \hat{\mathscr{E}} \mapsto \hat{\mathscr{C}}$. Then, $\hat{\pi}_*([s_z, 0]) = s_z \bmod \tau_f(H^0(\mathbb{P}^1, T_{\mathbb{P}^1})) \in T_{[f(\mathbb{P}^1)]}(\Sigma)$. The crucial observation is that all $\mathbb{C}$-linear combination of such s_z will also lie in $T_{[f(\mathbb{P}^1)]}(\Sigma)$. We will use them to generate holomorphic sections tangent to Σ and vanishing at two distinct points.

First we assert that if f^*T_X has at least two trivial components in the decomposition into holomorphic line bundles, then the criterion of Lemma (2.4.3) is automatically satisfied. In fact, in that case $H^0(\mathbb{P}^1, f^*T_X \otimes \mathscr{I}_0 \otimes \mathscr{I}_\infty)$ is of dimension at least three, while $\tau_f H^0(\mathbb{P}^1, T_{\mathbb{P}^1} \otimes \mathscr{I}_0 \otimes \mathscr{I}_\infty)$ is of dimension 1 so that there are at least two sections s_1, s_2 which vanish at 0 and ∞ and which are $\mathbb{C}$-linear independent mod $\tau_f H^0(\mathbb{P}^1, T_{\mathbb{P}^1} \otimes \mathscr{I}_0 \otimes \mathscr{I}_\infty)$. This means that there must exist at least one such s such that s mod $H^0(\mathbb{P}^1, T_{\mathbb{P}^1} \otimes \mathscr{I}_0 \otimes \mathscr{I}_\infty)$ is tangent to Σ, as Σ is of codimension 1 in $\hat{\mathscr{C}}$, if nonempty. Thus, the criterion in Lemma (2.4.3) is automatic.

Thus, for a generic special rational curve, we have either $f^*T_X \cong \mathscr{O}(3) \oplus \mathscr{O}(1)^{n-2} \oplus \mathscr{O}$ or $f^*T_X \cong \mathscr{O}(2) \oplus \mathscr{O}(2) \oplus \mathscr{O}(1)^{n-3} \oplus \mathscr{O}$. In either case one can show that if $\Theta|_{\hat{\mathscr{E}} - \mathscr{V}}$ is not of maximal rank, then the criterion of Lemma (2.4.3) is satisfied. However, we are going to prove the complete statement of Proposition (2.3) by ruling out the former possibility. If $f^*T_X \cong \mathscr{O}(3) \oplus \mathscr{O}(1)^{n-2} \oplus \mathscr{O}$ then $f\colon \mathbb{P}^1 \mapsto X$ must be ramified since any nontrivial bundle homomorphism $\nu\colon \mathscr{O}(2) \cong T_{\mathbb{P}^1} \mapsto f^*T_X \cong \mathscr{O}(3) \oplus \mathscr{O}(1)^{n-2} \oplus \mathscr{O}$ is degenerate at a unique point. Now let f be ramified at 0, say (the point being unique). Suppose without loss of generality that $n \geqslant 3$. (If $n \leqslant 2$ Proposition (2.3) is valid because of Howard-Smyth [11].) For $n \geqslant 3$, by counting dimensions one knows that there exists some $[f(\mathbb{P}^1)] \in \Sigma$, a 1-dimensional family $\mathscr{F}_\Gamma = \{[f_t] = t \in \Gamma \subset \mathscr{H}\}$ of rational curves $f_t\colon \mathbb{P}^1 \to X$, $\mathscr{O} \in \Gamma$, such that $f_t(0) = f(0) = x$, $f_0 = f$, $[f_t(\mathbb{P}^1)] \in \Sigma$ and $\mathscr{F}_\Gamma$ intersects the $\operatorname{Aut}(\mathbb{P}; 0)$-orbit $\{f \circ \varphi\colon \varepsilon \in \operatorname{Aut}(\mathbb{P}; 0)\}$ in $\mathscr{H}$ transversally. Let θ be a tangent vector of Γ at 0. Identify θ as an element of $H^0(\mathbb{P}^1, f^*T_X \otimes \mathscr{I}_0)$. Recall that f is ramified precisely at 0. If the vanishing order of θ at 0 is precisely one then one sees by expanding in local coordinates that f_γ is unramified for $\gamma \in \Gamma$ sufficiently close to 0, contradicting the fact that every generic special rational curve is ramified. So, $\theta \in H^0(\mathbb{P}^1, f^*T_X \otimes \mathscr{I}_0^2)$. However, since $f^*T_X \cong \mathscr{O}(3) \oplus \mathscr{O}^{n-2}(1) \oplus \mathscr{O}$, every such θ comes from $\tau_f H^0(\mathbb{P}^1, T_{\mathbb{P}^1} \otimes \mathscr{I}_0)$, f being ramified at 0. This contradicts the transversality between $\mathscr{F}_\Gamma$ and $\{f \circ \varphi = \varphi \in \mathrm{Aut}(\mathbb{P}; 0)\}$ and rules out the possibility $f^*T_X \cong \mathcal{O}(3) \oplus \mathcal{O}(1)^{n-3} \oplus \mathcal{O}$ for generic rational curve.

We are now left with the situation where for a generic special rational curve $f\colon \mathbb{P}^1 \mapsto X$, $f^*T_X \cong \mathcal{O}(2)^2 \oplus \mathcal{O}(1)^{n-3} \oplus \mathcal{O}$. In this case all generic special rational curves $f\colon \mathbb{P}^1 \to X$ are unramified, and $T_{\mathbb{P}^1}$ can be regarded as a holomorphic direct summand of f^*T_X. Then $T_{[f(\mathbb{P}^1)]}(\hat{\mathscr{C}}) \cong H^0(\mathbb{P}^1, \mathcal{O}(2) \oplus \mathcal{O}(1)^{n-3} \oplus \mathcal{O})$. Suppose $\Theta|_{\hat{\mathscr{E}}_\Sigma - \mathscr{V}}$ is not of maximal rank. Let s_z be a section in $H^0(\mathbb{P}^1, \mathcal{O}(2) \oplus \mathcal{O}(1)^{n-3} \oplus \mathcal{O})$ vanishing at z to the second order. s_z is unique up to a scalar multiple. Suppose $[f(\mathbb{P}^1)]$ is a smooth point of Σ. Then $s_z \bmod \tau_f(H^0(\mathbb{P}^1, T_{\mathbb{P}^1})) \in T_{[f(\mathbb{P}^1)]}(\Sigma)$. By taking $\mathbb{C}$-linear combinations of s_z we conclude that $H^0(\mathbb{P}^1, \mathcal{O}(2)) \bmod \tau_f H^0(\mathbb{P}^1, T_{\mathbb{P}^1}) \subset T_{[f(\mathbb{P}^1)]}(\Sigma)$, where $\mathcal{O}(2)$ denotes a component of $f^*T_X \cong \mathcal{O}(2) \oplus \mathcal{O}(1)^{n-3} \oplus \mathcal{O}$. In particular, for any $z_1, z_2, z_1 \neq z_2$, there exist $0 \neq s \bmod \tau_f(H^0(\mathbb{P}, T_{\mathbb{P}^1})) \in T_{[f(\mathbb{P}^1)]}(\Sigma)$ such that s vanishes at z_1 and z_2, proving the criterion of Lemma (2.4.3). Since this criterion leads to a contradiction to the criterion of Mumford-Grauert on blowing down curves, we have proved that $\Theta|_{\hat{\mathscr{E}}_\Sigma - \mathscr{V}}$ is of maximal rank, completing the proof of Proposition (2.3).

3. Invariance of $\mathscr{S}$ under holonomy

(3.1). Let (X, h) be as in §2, $X \not\cong \mathbb{P}^n$, and $\mathscr{S} \subset \mathbb{P}T_X$ be constructed from deforming rational curves of minimal degree, as in Proposition (2.3). $\mathscr{S} \subset \mathbb{P}T_X$ is an irreducible complex-analytic subvariety by construction. For $x \in X$, let $H_x(X)$ denote the holonomy group of (X, h) at x. All $H_x(X)$ are isomorphic as Lie groups. Every simply-connected complete Riemannian manifold admits a de Rham decomposition into irreducible factors. If there is no flat factor, the decomposition is unique up to permutation. Furthermore, if the Riemannian manifold admits no flat factors and is Kähler, then the de Rham factors are also Kähler. Since our Kähler manifold X is simply-connected, compact, verifying $b_2(X) = 1$, it follows immediately that (X, h) is irreducible as a Riemannian manifold. In fact, if $(X, h) \cong (X_1, h_1) \times (X_2, h_2)$ with X_1, X_2 compact Kähler, then the Kähler forms of (X_i, h_i), $i = 1, 2$, would give rise to two $\mathbb{R}$-linearly independent classes of $H^2(X, \mathbb{R})$. For simply-connected irreducible complete Riemannian manifolds, we have

Theorem (*Berger* [4], *Simons* [21]). *Let (M, g) be an irreducible, simply-connected, complete Riemannian manifold. Then, either* (i) *for all $x \in M$, $H_x(M)$ acts transitively on the unit sphere of the tangent space at x; or* (ii) *(M, g) is an irreducible Riemannian symmetric space of rank $\geqslant 2$.*

Clearly, for a compact Kähler manifold (X, h), the second alternative would mean that (X, h) is a compact Hermitian symmetric space of rank $\geqslant 2$. Thus, to prove Theorem 1 and hence the Main Theorem, it suffices to show

Proposition (3.1). *Suppose X is not biholomorphic to $\mathbb{P}^n$ and $\mathcal{S} \subset \mathbb{P}T_X$ is constructed as above. Then $\mathcal{S}$ is invariant under parallel transport.*

Let $\gamma\colon (-\delta, \delta) \mapsto X$ be a curve on X with $\gamma(0) = x$, $\dot{\gamma}(t) \neq 0$ for $-\delta < t < \delta$. Let v be a real tangent vector at 0. Then the parallel transport of v along γ is the unit vector field $v(t)$ on γ such that $\nabla_{\dot{\gamma}(t)} v(t) \equiv 0$ and $v(0) = v$. Clearly the notion of parallel transport extends to complexified tangent vectors. On a Kähler manifold, tangent vectors α of type $(1,0)$ will be transformed to tangent vectors $\alpha(t)$ of the same type under parallel transport. Clearly, for any complex number c, the parallel transport of $c\alpha$ along γ is given by $c\alpha(t)$, so that we can talk about parallel transport of the complex line $\mathbb{C}\alpha$ along γ. Equivalently, the holonomy group $H_x(X)$ acts naturally on $\mathbb{P}T_x$. Proposition (3.1) asserts then that $\mathcal{S}_x \subset \mathbb{P}T_x$ is invariant under $H_x(X)$ under the induced action. We remark here that Proposition (3.1) implies that the second case of our construction of $\mathcal{S}$ does not correspond to a real situation. In fact, for an irreducible compact Hermitian symmetric space X, $q = n + 1$ if and only if $X \cong \mathbb{P}^n$, in which case there certainly exists no subset of $\mathbb{P}T_x$ invariant under holonomy. However, I do not know of any a priori method of ruling this case out.

In order to prove Proposition (3.1) we state first a lemma which says that $\mathcal{S}$ is invariant under parallel transport if and only if it is true infinitesimally. To start with we need some definitions. Recall the notion of generic rational curves and generic special rational curves $[f]$ in Part I and II respectively of Proposition (2.3). In the former case write $f^*T_X \cong \mathcal{O}(a_1) \oplus \cdots \mathcal{O}(a_{n-l}) \oplus \mathcal{O}^l$ with $a_1 \geqslant \cdots \geqslant a_{n-l} > 0$, $l \neq 0$. In the latter case we have $f^*T_X \cong \mathcal{O}(2)^2 \oplus \mathcal{O}(1)^{n-3} \oplus \mathcal{O}$. We will say that $[\alpha] \in \mathcal{S}_x$ is a generic point of $\mathcal{S}$ if there exists a generic rational curve (respectively a generic special rational curve) f: $P^1 \mapsto X$ such that $f(0) = X$, f is unramified at 0, and $df(\frac{\partial}{\partial z}) = \alpha$ for a local holomorphic parameter z at 0. In this case we have $T_{[\alpha]}(\mathcal{S}_x) = V_x/\mathbb{C} \subset T_{[\alpha]}(\mathbb{P}T_x)$, where $V_x \subset T_x(X)$ corresponds to the direct sum of positive components in the line bundle decomposition of f^*T_X. The proof we gave in Part I of Proposition (2.3) extends in an obvious way to the case of generic special rational curves of Part II, given, in the terminology of (2.4), $\theta|_{\hat{\mathcal{E}}_\Sigma - \mathcal{V}}$ is of maximal rank. Clearly, $T_{[\alpha]}(\mathcal{S}_X) \subset V_x/\mathbb{C}\alpha$. One has in fact $T_{[\alpha]}(\mathcal{S}_x) = V_x/\mathbb{C}\alpha$ by counting dimensions since $\dim_{\mathbb{C}} T_{[\alpha]}(\mathcal{S}_x) = \dim V_x/\mathbb{C}\alpha = n - 2$. It is clear that for a generic $[\alpha]$ of $\mathcal{S}$, $[\alpha] \in \mathcal{S}_x$, $[\alpha]$ is a smooth point of both $\mathcal{S}_x$ and $\mathcal{S}$. We now state

Definition. Let $[\alpha] \in \mathcal{S}_x$ be a generic point of $\mathcal{S}$. We say that $\mathcal{S}$ is infinitesimally invariant under parallel transport at $[\alpha]$ if for any smooth curve

γ: $(-\delta, \delta) \mapsto X$ verifying $\gamma(0) = x$ and $\dot\gamma(0) = \eta \neq 0$, and for any smooth mapping α: $(-\delta, \delta) \mapsto T_X$ verifying $[\alpha(t)] \in \mathscr{S}_{\gamma(t)}$, we have $\nabla_\eta \alpha(0) \in V_x$. We will say that $\mathscr{S}$ is infinitesimally invariant under parallel transport if it is so at every generic point $[\alpha]$ of $\mathscr{S}$.

We can now state our lemma

Lemma (3.1). *Suppose $\mathscr{S}$ is infinitesimally invariant under parallel transport. Then it is invariant under holonomy in the sense that $\mathscr{S}_x$ is invariant under the induced action of $H_x(X)$ for all $x \in X$.*

Proof of Lemma (3.1). Let $[\alpha] \in \mathscr{S}_x$ be a generic point of $\mathscr{S}$. Let γ: $(-\delta, \delta) \mapsto X$ be a smooth curve verifying $\gamma(0) = 0$, $\dot\gamma(t) = \eta_t \neq 0$, $-\delta < t < \delta$. Let α_t: $(-\delta, \delta) \mapsto T_X$ be obtained from $\alpha_0 = \alpha$ by parallel transport along γ, i.e. $\alpha_t \in T_{\gamma(t)}(X)$ and $\nabla_{\gamma(t)}\alpha_t \equiv 0$ on γ. We assert that $[\alpha_t] \in \mathscr{S}_{\gamma(t)}$. This will prove the lemma. In fact, let λ: $[0,1]$ be any smooth function (up to end points) on X such that $\lambda(0) = \lambda(1) = y \in X$ and let $[\beta] \in \mathscr{S}_y$. If $[\beta]$ is a generic point of $\mathscr{S}$ and λ is real-analytic up to end points, then since our assertion implies $[\beta_t] \in \mathscr{S}_{\lambda(t)}$ for t sufficiently small it follows immediately that for all t, $0 \leqslant t \leqslant 1$, $[\beta_t] \in \mathscr{S}_{\lambda(t)}$ from the identity theorem of real analytic functions. Since any smooth curve can be approximated by real-analytic curves it follows that the same is true for smooth loops λ, whenever $[\beta] \in \mathscr{S}_y$ is a generic point of $\mathscr{S}$. Since the set of generic points $[\beta]$ of $\mathscr{S}$ is dense in $\mathscr{S}$, it follows immediately that $\mathscr{S}$ is invariant under holonomy (and as a consequence that $\mathscr{S}$ is nonsingular). To prove the assertion $[\alpha_t] \in \mathscr{S}_{\gamma(t)}$ let, without loss of generality, $\|\alpha\| = 1$. Write $S \subset T_X$ for the set of all unit vectors β such that $[\beta] \in \mathscr{S}$. Define r: $(-\delta, \delta) \mapsto \mathbb{R}$ by $r(t) = d(\alpha_t; \mathscr{S})$ where d is the Euclidean distance in terms of some local Euclidean coordinates of $\mathscr{S}$ at α. Shrinking δ if necessary, we may assume that r is well defined and that r^2 is smooth on $(-\delta, \delta)$. It is immediate from the definition of infinitesimal invariance under parallel transport to deduce that α_t is tangent to $\mathscr{S}$ at α. Suppose $\beta_t \in \mathscr{S}_{\gamma(t)}$ is a point of $\mathscr{S}_{\gamma(t)}$ such that $d(\alpha_t, \mathscr{S}_{\gamma(t)}) = d(\alpha_t, \beta_t)$. Then, to transport α_t in a parallel way along γ starting at $\gamma(t)$ it is equivalent to take the sum of the parallel transports of β_t and $\alpha_t - \beta_t$. For t sufficiently small, $[\beta_t]$ is also a generic point of $\mathscr{S}$ (the set of generic points being open). We see immediately that there exists a positive constant C such that

$$\frac{d}{dt}r(t) < Cr(t) \quad \text{for } r(t) \neq 0,\ r(0) = 0,$$

$$\frac{d}{dt}r^2(t) < 2Cr^2(t).$$

Thus, for any $t_0 > 0$ sufficiently small,

$$\log r^2(t) \leqslant \log r^2(t_0) + 2C(t - t_0) \quad \text{for } t_0 \leqslant t \leqslant \delta.$$

Letting $t_0 \mapsto 0$ we see immediately that $r \equiv 0$ (similarly for $-\delta < t \leqslant 0$). This proves the lemma.

(3.2) The infinitesimal invariance under parallel transport. To prove Proposition (3.1) it remains to show

Proposition (3.1)′. *$\mathscr{S} \subset \mathbb{P}T_X$ is infinitesimally invariant under parallel transport.*

Proof. Let $[\alpha_0] \in \mathscr{S}_x$ be a generic point of $\mathscr{S}$. Let $\gamma\colon (-\delta, \delta) \mapsto X$ be a smooth curve on X such that $\gamma(0) = x$ and $\dot\gamma(0) = \eta \neq 0$. In this section we write $\beta(t)$, $-\delta < t < \delta$, $\beta(t) \in T_{\gamma(t)}$, for the parallel transport of α along γ (in place of the notation α_t). Thus $\beta(0) = \alpha_0$ and $\nabla_{\gamma(t)}\beta \equiv 0$. Let $\alpha(t) \in T_{\gamma(t)}$ be such that $\alpha(0) = \alpha_0$, $[\alpha(t)] \in \mathscr{S}_{\gamma(t)}$, and $\alpha\colon (-\delta, \delta) \mapsto T_X$ is smooth. At each $\gamma(t)$, let $N_{\gamma(t)}$ be the null-space of the Hermitian bilinear form $H_{\alpha(t)}$ defined by $H_{\alpha(t)}(\xi, \eta) = R_{\alpha(t)\overline{\alpha(t)}\xi\bar\eta}$. Let $\mathscr{D}_{\gamma(t)}$ be the orthogonal complement of $\mathscr{N}_{\gamma(t)} + \mathbb{C}\alpha$ in $T_{\gamma(t)}(X)$. We are going to prove

(I) $\nabla_\eta \alpha(0) \in \mathbb{C}\alpha + \mathscr{D}_{\gamma(0)} = \mathscr{N}^{\perp}_{\gamma(0)}$.

(II) $\mathscr{N}^{\perp}_{\gamma(0)} \subset \mathscr{V}_{\gamma(0)}$.

Proof of (I). Write

$$(1) \qquad \beta(t) = \alpha(t) + t\xi(t) + t\zeta(t), \qquad \xi(t) \in \mathscr{D}_{\gamma(t)},\ \zeta(t) \in \mathscr{N}_{\gamma(t)}.$$

We have by the definition of β

$$(2) \qquad 0 = \nabla_\eta \beta(0) = \nabla_\eta \alpha(0) + \xi + \zeta, \qquad \xi = \xi(0),\ \zeta = \zeta(0).$$

To prove (I) it suffices therefore to show that $\zeta = 0$. Consider the vector field $\chi(t)$ along γ obtained by parallel transport of ζ, i.e. $\chi(0) = \zeta$ and $\nabla_{\gamma(t)}\chi \equiv 0$. Write

$$(3) \qquad \chi(t) = \zeta'(t) + t\theta(t), \qquad \zeta'(t) \in \mathscr{N}_{\gamma(t)},\ \theta(t) \in \mathscr{N}^{\perp}_{\gamma(t)}.$$

Consider now the expansion

$$(4) \qquad R_{\beta(t)\overline{\beta(t)}\chi(t)\overline{\chi(t)}} = t^2 A + O(t^3).$$

Using results from §1, we have

$$(5) \quad A = \left(R_{\alpha\bar\alpha\theta\bar\theta} + R_{\xi\bar\xi\zeta\bar\zeta} + 2\,\mathrm{Re}R_{\alpha\bar\theta\zeta\bar\xi} + 2\,\mathrm{Re}\,R_{\alpha\bar\zeta\theta\bar\xi}\right) + R_{\zeta\bar\zeta\zeta\bar\zeta} + 2\,\mathrm{Re}\,R_{\xi\bar\zeta\zeta\bar\zeta}$$
$$= B + R_{\zeta\bar\zeta\zeta\bar\zeta} + 2\,\mathrm{Re}\,R_{\xi\bar\zeta\zeta\bar\zeta}.$$

Consider the expansion

$$R\left(\alpha + t\xi,\ \overline{\alpha + t\xi},\ \zeta + t\theta,\ \overline{\zeta + t\theta}\right).$$

then B is the coefficient of t^2 in the expansion. Since $R \geqslant 0$ and $R_{\alpha\bar\alpha\zeta\bar\zeta} = 0$, it follows that $B \geqslant 0$. Thus,

$$(6) \qquad A \geqslant R_{\zeta\bar\zeta\zeta\bar\zeta} + 2\,\mathrm{Re}\,R_{\xi\bar\zeta\zeta\bar\zeta}.$$

We assert that $R_{\xi\bar\zeta\zeta\bar\zeta} = 0$. For this we are going to make use of structural equations on the curvature tensor arising from evolution equations. Recall that

$$(7)\qquad 0 \leqslant \Delta R_{\alpha\bar{\alpha}\zeta\bar{\zeta}} = \frac{\partial}{\partial t} R_{\alpha\bar{\alpha}\zeta\bar{\zeta}} - F(R)_{\alpha\bar{\alpha}\zeta\bar{\zeta}}.$$

Here we are considering some evolved metric $h_{i\bar{j}} = g_{i\bar{j}}(t_0)$, $t_0 > 0$, and Δ is taken with respect to h. Since we know that $(X, g_{i\bar{j}}(t))$ carries nonnegative holomorphic bisectional curvature for $0 \leqslant t <$ some ε, taking $0 < t_0 < \varepsilon$, we see that $\frac{\partial}{\partial t} R_{\alpha\bar{\alpha}\zeta\bar{\zeta}} = 0$ at $t = t_0$. (Recall $R_{\alpha\bar{\alpha}\zeta\bar{\zeta}}(t_0) = 0$.) Moreover by §1, (1.2) we know that at $t = t_0$.

$$(8)\qquad F(R)_{\alpha\bar{\alpha}\zeta\bar{\zeta}} \geqslant 0.$$

Combined with (7) this forces

$$(9)\qquad \Delta R_{\alpha\bar{\alpha}\zeta\bar{\zeta}} = 0,$$

$$(10)\qquad F(R)_{\alpha\bar{\alpha}\zeta\bar{\zeta}} = 0.$$

We are going to derive $R_{\xi\bar{\xi}\zeta\bar{\zeta}} = 0$ for $\xi \in \mathcal{N}^{\perp}_{\gamma(t)}$ and $\zeta \in \mathcal{N}_{\gamma(t)}$ by using (10). As in §1, equation (12), let $\{e_\mu\}$ be an orthonormal basis of $T(X)$ consisting of eigenvectors of H_α so that $R_{\alpha\bar{\alpha}1\bar{1}} \geqslant R_{\alpha\bar{\alpha}2\bar{2}} \geqslant \cdots \geqslant R_{\alpha\bar{\alpha}m\bar{m}} > 0$ and $R_{\alpha\bar{\alpha}l\bar{l}} = 0$ for $l > m$. Recall the inequality (13) of §1 for $\chi = e_\mu$ for $\mathcal{S}_{\nu,\mu} = R_{\alpha\bar{\mu}\zeta\bar{\nu}} + R_{\alpha\bar{\zeta}\nu\bar{\mu}}$. Namely

$$(11)\qquad R_{\mu\bar{\mu}\zeta\bar{\zeta}} \geqslant \sum_{1 \leqslant \nu \leqslant m} \frac{|\mathcal{S}_{\nu,\mu}|^2}{R_{\alpha\bar{\alpha}\nu\bar{\nu}}}.$$

However, since $F(R)_{\alpha\bar{\alpha}\zeta\bar{\zeta}} \geqslant \Sigma_{\sigma,\tau} |R_{\alpha\bar{\zeta}\sigma\bar{\tau}}|^2$ and $F(R)_{\alpha\bar{\alpha}\zeta\bar{\zeta}} = 0$ by (9), we have $R_{\alpha\bar{\zeta}\nu\bar{\mu}} = 0$ and thus

$$(12)\qquad R_{\mu\bar{\mu}\zeta\bar{\zeta}} \geqslant \sum_{1 \leqslant \nu \leqslant m} \frac{|R_{\alpha\bar{\mu}\zeta\bar{\nu}}|^2}{R_{\alpha\bar{\alpha}\nu\bar{\nu}}}.$$

Recall the inequality $F(R)_{\alpha\bar{\alpha}\zeta\bar{\zeta}} \geqslant 0$ was obtained by adding the inequalities (13) of §1 (reduced to (12) here using (10)) with $\chi = e_\mu$. Equation (10), i.e. $F(R)_{\alpha\bar{\alpha}\zeta\bar{\zeta}} = 0$, then forces (12) to be an equality for all μ, $1 \leqslant \mu \leqslant m$, i.e.

$$(13)\qquad R_{\mu\bar{\mu}\zeta\bar{\zeta}} = \sum_{1 \leqslant \nu \leqslant m} \frac{|R_{\alpha\bar{\mu}\zeta\bar{\nu}}|^2}{R_{\alpha\bar{\alpha}\nu\bar{\nu}}}.$$

Now inequality (11) is valid for any χ in place of the basis vector e_μ. Moreover $\mathcal{S}_{\nu,\mu}$ is replaced by $\mathcal{S}_{\nu,\chi} = R_{\alpha\bar{\chi}\zeta\bar{\nu}} + R_{\alpha\bar{\zeta}\chi\bar{\nu}} = R_{\alpha\bar{\chi}\zeta\bar{\nu}}$. Consider now the special case of $\chi = e_\mu + \varepsilon\bar{\zeta}$. Then $\mathcal{S}_{\nu,\chi} = R_{\alpha\bar{\mu}\zeta\bar{\nu}} + \varepsilon R_{\alpha\bar{\zeta}\zeta\bar{\nu}} = R_{\alpha\bar{\mu}\zeta\bar{\nu}} = \mathcal{S}_{\nu,\mu}$ and we have the inequality

$$(14)\qquad R\left(e_\mu + \varepsilon\zeta, \overline{e_\mu + \varepsilon\zeta}\,; \zeta, \bar{\zeta}\right) \geqslant \sum_{1 \leqslant \nu \leqslant m} \frac{|R_{\alpha\bar{\mu}\zeta\bar{\nu}}|^2}{R_{\alpha\bar{\alpha}\nu\bar{\nu}}} = R_{\mu\bar{\mu}\zeta\bar{\zeta}},$$

for any real ε. From (14) we obtain the variation equality

$$\frac{\partial}{\partial \varepsilon} R\left(e_\mu + \varepsilon\zeta,\ \overline{e_\mu + \varepsilon\zeta}\ ;\ \zeta, \bar{\zeta}\right)\Big|_{\varepsilon=0} = 0, \tag{15}$$

since the function under consideration attains its minimum at $\varepsilon = 0$. Expanding (15) we obtain

$$2\,\mathrm{Re}\, R_{\mu\bar{\zeta}\zeta\bar{\zeta}} = 0. \tag{16}$$

Clearly one can replace e_μ by $e^{i\theta}e_\mu$ for any real θ without changing the preceding argument. As a consequence, we obtain

$$R_{\mu\bar{\zeta}\zeta\bar{\zeta}} = 0 \quad \text{for } 1 \leqslant \mu \leqslant m,$$

and thus

$$R_{\xi\bar{\zeta}\zeta\bar{\zeta}} = 0 \quad \text{for } \xi \in \mathcal{N}^{\perp}_{\gamma(t)} = \sum_{\mu=1}^{m} \mathbf{C} e_\mu. \tag{17}$$

Now equation (17) implies by (6)

$$A \geqslant R_{\zeta\bar{\zeta}\zeta\bar{\zeta}}. \tag{18}$$

Recall by (3) that $A = (d/dt^2)R(\beta(t), \overline{\beta(t)};\ \chi(t), \overline{\chi(t)})$, $\beta(0) = \alpha$, $\chi(0) = \zeta$, and $\nabla_{\dot{\gamma}(t)}\beta(t) \equiv \nabla d\dot{\gamma}(t)\chi(t) \equiv 0$. Thus, for $\eta = \dot{\gamma}(0)$

$$\nabla_s^2 \eta R_{\alpha\bar{\alpha}\zeta\bar{\zeta}} = A \geqslant R_{\zeta\bar{\zeta}\zeta\bar{\zeta}}. \tag{19}$$

On the other hand, equation (9), i.e., $\Delta R_{\alpha\bar{\alpha}\zeta\bar{\zeta}} = 0$, and $\nabla_v^2 R_{\alpha\bar{\alpha}\zeta\bar{\zeta}} \geqslant 0$ for all real tangent vector v at $\gamma(t)$ imply

$$\nabla_\eta^2 R_{\alpha\bar{\alpha}\zeta\bar{\zeta}} = 0, \tag{20}$$

so that (19) forces

$$R_{\zeta\bar{\zeta}\zeta\bar{\zeta}} = 0. \tag{21}$$

However, as was proved in §1, Proposition (1.1), (X, h) carries positive holomorphic sectional curvature. It follows immediately from (21) that

$$\zeta = 0, \tag{22}$$

and thus

$$\nabla_\eta \alpha(0) = -\xi - \zeta = -\xi \in \mathbf{C}\alpha + \mathscr{D}_{\gamma(0)} = \mathcal{N}^{\perp}_{\gamma(0)}. \tag{23}$$

The proof of (I) is complete.

Proof of (II). We are going to show $\mathcal{N}^{\perp}_{\gamma(0)} \subset \mathscr{V}_{\gamma(0)}$, where $V_{\gamma(0)} \subset T_{\gamma(0)}$ corresponds to the positive components of the line bundle decomposition $f^*T_X \cong \mathcal{O}(a_1) \oplus \cdots \oplus \mathcal{O}(a_l) \oplus \mathcal{O}^{n-l}$, $a_1 \geqslant \cdots \geqslant a_l > 0$, for a generic (respectively generic special rational curve) rational curve $f\colon \mathbb{P}^1 \mapsto X$ verifying

$f(0) = \gamma(0)$, and $df(T_0(\mathbb{P}^1)) = \mathbb{C}\alpha$. (In the latter case $a_1 = a_2 = 2$; $a_3 = \cdots a_{n-1} = 1$.) We are going to relate holomorphic bisectional curvature and the line bundle decomposition of f^*T_X. Let Θ denote the curvature tensor of the dual cotangent bundle T_X^* in terms of the metric induced by h. We assert

$(*)$ For any $\mu^* \in T^*_{\gamma(0)}$ such that $\mu^*(V_{\gamma(0)}) = 0$, we have $\Theta_{\mu^*\bar{\mu}^*\alpha\bar{\alpha}} = 0$.

To prove $(*)$, consider first of all the direct sum decomposition

$$f^*(T_X^*) \cong \mathcal{O}(-a_1) \oplus \cdots \oplus \mathcal{O}(-a_l) \oplus \mathcal{O}^{n-l}.$$

Write $f^*T_X = V \oplus W$ for the decomposition into the positive and the trivial parts, i.e. $V \cong \Sigma_{i=1}^l \mathcal{O}(a_i)$, $W \cong \mathcal{O}^{n-l}$. Write $W^{\#}$ for the annihilator of W, etc., i.e.

$$W_z^{\#} = \left\{ \mu^* \in \left(f^*(T_X^*)\right)_z \,|\, \mu^*(w) = 0 \text{ for all } w \cap W_z \right\}.$$

Then the decomposition of $f^*(T_X^*)$ corresponds to $f^*(T_X^*) = W^{\#} \oplus V^{\#}$, with $W^{\#} \cong \Sigma_{i=1}^l \mathcal{O}(-a_i)$ and $V^{\#} \cong \mathcal{O}^{n-l}$. Let $\Theta^{V^{\#}}$ denote the curvature of the holomorphic vector subbundle $V^{\#} \subset f^*(T_X^*)$. Identifying $(f^*T_X^*)_0$ with $T^*_{\gamma(0)}(X)$, and writing $V^{\#}_{\gamma(0)} \subset T^*_{\gamma(0)}(X)$ for the $\mathbb{C}$-subspace corresponding to $V_0^{\#}$, etc., we have

$$\Theta^{V^{\#}}_{\mu^*\bar{\mu}^*\alpha\bar{\alpha}} \leqslant \Theta_{\mu^*\bar{\mu}^*\alpha\bar{\alpha}} \quad \text{for } \mu^* \in V^{\#}_{\gamma(0)}.$$

However, since $V^{\#} \cong \mathcal{O}^{n-l}$ and $\Theta_{\mu^*\bar{\mu}^*\alpha\bar{\alpha}} \leqslant 0$ because (X, h) carries nonnegative holomorphic bisectional curvature, the preceding inequality forces

$$\Theta^{V^{\#}}_{\mu^*\bar{\mu}^*\alpha\bar{\alpha}} = \Theta_{\mu^*\bar{\mu}^*\alpha\bar{\alpha}} = 0 \quad \text{for } \mu^* \in V^{\#}_{\gamma(0)}.$$

This proves assertion $(*)$.

Recall that if $\mu \in T_{\gamma(0)}(X)$ is such that μ and μ^* correspond to each other under the contraction by the Kähler metric, i.e., in local holomorphic coordinates

$$\begin{cases} \mu = \sum_i \mu^i \dfrac{\partial}{\partial z_i}, \\ \mu^* = \sum_{i,j} \left(h_{i\bar{j}}\overline{\mu^j}\right) dz^i, \end{cases} \quad \text{or} \quad \begin{cases} \mu = \sum_{i,j} \left(h^{i\bar{j}}\bar{\mu}_j\right) \dfrac{\partial}{\partial z_i}, \\ \mu^* = \sum_i \mu_i \, dz^i, \end{cases}$$

then we have $\Theta_{\mu^*\bar{\mu}^*\alpha\bar{\alpha}} = -R_{\mu\bar{\mu}\alpha\bar{\alpha}}$.

However, under the correspondence by contraction, $\mu^*(V_{\gamma(0)}) = 0$ if and only if

$$\sum_i \mu_i v^i = 0 \quad \text{for all } v = \sum_i v^i \frac{\partial}{\partial z_i} \in V_{\gamma(0)},$$

i.e.,

$$\sum_{i,j} h_{i\bar{j}} \overline{\mu^j} v^i = 0 \quad \text{for all such } v,$$

i.e. $\langle v, \mu \rangle = 0$ for all such v. Thus $\mu^*(V_{\gamma(0)}) = 0$ if and only if $\mu \perp V_{\gamma(0)}$. Hence we have obtained

$$R_{\alpha\bar{\alpha}\mu\bar{\mu}} = 0 \quad \text{whenever } \mu \perp V_{\gamma(0)}.$$

It follows that $\mathscr{N}_{\gamma(0)} \supset \mathscr{V}_{\gamma(0)}^{\perp}$, i.e., $\mathscr{N}_{\gamma(0)}^{\perp} \subset \mathscr{V}_{\gamma(0)}$, proving (II). The proof of Proposition (3.1) and hence of the Main Theorem is complete.

Added in proof. In both Mok-Zhong [17] and the present article the proofs rely on the remarkable vanishing of many curvature terms $R_{i\bar{j}k\bar{l}}$ in terms of a basis consisting of eigenvectors of the Hermitian form $H_\alpha(\xi, \eta) = R_{\alpha\bar{\alpha}\xi\bar{\eta}}$ associated to some vector α of type (1.0). We are led to guess the vanishing of such terms from a very special property of the curvature tensors of Hermitian symmetric spaces X: Let α be any tangent vector of type $(1,0)$ at $x \in X$ and let $\{e_i\}$ be a basis of $T_x^{1,0}(X)$ consisting of eigenvectors of the Hermitian form H_α. Then $R_{i\bar{j}k\bar{l}} = 0$ unless $R_{\alpha\bar{\alpha}i\bar{i}} + R_{\alpha\bar{\alpha}k\bar{k}} = R_{\alpha\bar{\alpha}j\bar{j}} + R_{\alpha\bar{\alpha}l\bar{l}}$. This property of Hermitian symmetric spaces follows from $\nabla R \equiv 0$ and from computing the commutation $0 = R_{i\bar{j}k\bar{l},\alpha\bar{\alpha}} - R_{i\bar{j}k\bar{l},\bar{\alpha}\alpha} = R_{i\bar{j}k\bar{l}}(R_{\alpha\bar{\alpha}i\bar{i}} + R_{\alpha\bar{\alpha}k\bar{k}} - R_{\alpha\bar{\alpha}j\bar{j}} - R_{\alpha\bar{\alpha}l\bar{l}})$. As an example see equation (17) in the proof of Proposition (3.1)′ of the present article.

References

[1] S. Bando, *On three-dimensional compact Kähler manifolds of nonnegative bisectional curvature*, J. Differential Geometry **19** (1984) 283–297.

[2] W. Barth, C. Peters & A. van de Ven, *Compact complex surfaces*, Ergebnisse Math. Grenzgeb., 3. Folge, Band 4, Springer, Berlin, 1984.

[3] A. Beauville, *Surfaces algebriques complexes*, Astérisque, No. 54, Soc. Math. France, Paris, 1978.

[4] M. Berger, *Sur les groupes d'holonomie homogènes des variétés a connexion affine et des variétés riemanniennes*, Bull. Soc. Math. France **83** (1955) 279–330.

[5] H.-D. Cao & B. Chow, *Compact Kähler manifolds with nonnegative curvature operator*, preprint.

[6] W.-L. Chow & B. L. van der Waerden, *Zur algebraischen Geometrie*. IX, Math. Ann. **113** (1937) 692–704.

[7] H. Grauert, *Über Modifikationen und exzeptionelle analytische Mengen*, Math. Ann. **146** (1962) 331–368.

[8] R. S. Hamilton, *Three-manifolds with positive Ricci curvature*, J. Differential Geometry **17** (1982) 255–306.

[9] ______, *Four-manifolds with positive curvature operator*, J. Differential Geometry **24** (1986) 153–179.

[10] H. Holmann, *Komplexe Räume mit Komplexen Transformationsgruppen*, Math. Ann. **150** (1963) 327–360.

[11] A. Howard & B. Smyth, *Kähler surfaces of nonnegative curvature*, J. Differential Geometry **5** (1971) 491–502.

[12] A. Howard, B. Smyth & H. Wu, *On compact Kähler manifolds of nonnegative bisectional curvature*. I, Acta Math. **147** (1981) 51–56.

[13] B. Kaup, *Über offene analytische Äquivalenzrelationen auf komplexen Räumen*, Math. Ann. **183** (1969) 6–16.

[14] K. Kodaira, *On the stability of compact submanifolds of complex manifolds*, Ann. of Math. (2) **85** (1963) 79–94.

[15] D. Mumford, *The topology of normal singularities of an algebraic surface and a criterion for simplicity*, Inst. Hautes Études Sci. Publ. Math. **9** (1961).

[16] N. Mok, *Uniqueness theorems of Kähler metrics of semipositive bisectional curvature on compact Hermitian symmetric spaces*, preprint.

[17] ______, *Compact Kähler manifolds of nonnegative holomorphic bisectional curvature*, Complex Analysis and Algebraic Geometry, Proc. Göttingen 1985, ed. H. Grauert, Lecture Notes in Math., Vol 1194, Springer, Berlin, 90–103.

[18] N. Mok & J.-Q. Zhong, *Variétés compactes kähleriennes d'Einstein de courbure bisectionnelle semipositive*, C. R. Acad. Sci. Paris Sér. I **296** (1985) 473–475.

[19] ______, *Curvature characterization of compact Hermitian symmetric spaces*, J. Differential Geometry **23** (1986) 15–67.

[20] S. Mori, *Projective manifolds with ample tangent bundles*, Ann. of Math. (2) **110** (1979) 593–606.

[21] J. Simons, *On the transitivity of holonomy systems*, Ann. of Math. (2) **76** (1962) 213–234.

[22] Y.-T. Siu, *Curvature characterization of hyperquadrics*, Duke Math. J. **47** (1980) 641–654.

[23] ______, *Complex analyticity of harmonic maps, vanishing and Lefschetz theorems*, J. Differential Geometry **17** (1982) 55–138.

[24] Y.-T. Siu & S.-T. Yau, *Compact Kähler manifolds of positive bisectional curvature*, Invent. Math. **59** (1980) 189–204.

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